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The Mathematics of the Physical Properties of Crystals

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SECTION 1

Introduction

THE use of crystals as oscillating elements and as light valves in electric circuits has given the mathematics of crystalline media an engineering importance. Soon after the first simple quartz oscillators were made it was noticed that some ways of cutting the block from the natural crystal gave lower temperature coefficients of frequency than did other ways. This led to studies of the change of elastic modulii with direction and temperature and finally to the discovery that there are directions in quartz for which the shear modulus does not change with temperature.

Such computations are rather involved, and there is, in the English language, no general reference book on these new problems. The existing works were evidently not written with the idea in mind that anyone would ever actually do much numerical work with directional properties of crystals, since the methods used are not the best suited to this. The matrix algebra has the advantages of a symbolic algebra and is also, through the concept of matrix multiplication, a scheme for computing numerical results.

As the problem of temperature coefficients of frequency involves the temperature coefficient of expansion, the temperature coefficient of density and the temperature coefficient of elastic modulii, these problems must be put into the language of matrix algebra so that they will fit into the general structure being built for more difficult problems. For this reason, after an introduction to the idea of linear vector functions, through consideration of the relation between the electric field and the induction in a crystal, and a hasty sketch of symmetry types found in crystals, we proceed to the consideration of stress and strain and their relations to each other.

Following these, we take up piezo electricity and the converse piezo electric effects; these are important as they tell us the ways a crystal may be driven. We have not seen anywhere a general proof that the modulii of the converse effect are the same numbers as the modulii of the direct effect —to the first order of small quantities, though Lippman predicted the converse effect and demonstrated its magnitude to be about this; he ap-

parently didn't consider the general case of six simultaneous stress components, six strain components, three electric field components and three induction components. The fact that the mentioned relation is true only to the first order of small quantities seems to have escaped the attention of some experimenters who have sought to show non-linearity of the piezoelectric effect by demonstrating non-linearity in the converse effect.

As a basis for light valve problems, we handle the propagation of light through crystals, then the electro optic effect and the piezo optic effect.

SECTION 2

A Linear Vector Function

For almost every physical constant of an isotropic medium a crystalline medium has several constants. For instance, a piece of glass has a coefficient of thermal expansion but a crystal has many coefficients of thermal expansion, the coefficient depending on direction. It might be thought that there were no necessary relations between the coefficients in different directions but there are necessary relations.

As an example of the simplifying relations between the values of physical constants in different directions let us consider a crystal in an electric field. Measurements show that the dielectric constant varies with direction in a crystal. If the field is not in the direction of greatest dielectric constant, the displacement current might veer over a little, much as a nail tries to follow the grain of the wood. We shall assume that for any electric field vector E there corresponds an electric induction vector D which may not coincide with E. Also we assume that the magnitude of D is proportional to E, that is, if E results in D, then nE results in nD. Lastly, we assume that if E_1 results in D_1, E_2 in $D_2 \ldots$ and E_m results in D_m, then $E_1 + E_2 \ldots E_m$ results in $D_1 + D_2 + \ldots D_m$. If these assumptions hold, then as any arbitrary field E can be expressed as the vector sum of its three components E_1, E_2, E_3 along three arbitrary unit vectors i, j, k, the induction vector resulting from E can be computed from the induction vectors resulting from E_1, E_2, and E_3. For, let E_1 result in $D_{11}i + D_{21}j + D_{31}k$, E_2 result in $D_{12}i + D_{22}j + D_{32}k$ and E_3 result in $D_{13}i + D_{23}j + D_{33}k$, then $E_1 + E_2 + E_3 = E$ results in the induction vector:

$$\begin{aligned} D = \quad & (E_1D_{11} + E_2D_{12} + E_3D_{13})i \\ & + (E_1D_{21} + E_2D_{22} + E_3D_{23})j \\ & + (E_1D_{31} + E_2D_{32} + E_3D_{33})k \end{aligned} \qquad \ldots\ldots\ldots\ldots (2.1)$$

It is seen then, that not more than 9 constants are needed to describe the dielectric properties of a crystal. The energy required to establish the

electric field is half the product of the component of the induction in the direction of the field and the electric field. This is, therefore:

$$2W = E_1^2D_{11} + E_2^2D_{22} + E_3^2D_{33} + E_2E_3(D_{23} + D_{32}) + E_3E_1(D_{31} + D_{13}) + E_1E_2(D_{12} + D_{21})$$

Considering then a condenser made from a unit cube of crystal, the charge is D and the energy content is W. If there is no leakage loss, the charge that can be drawn from the condenser is $D = \frac{\partial W}{\partial E}$. Whence $D_1 = \frac{\partial W}{\partial E_1} = D_{11}E_1 + \frac{1}{2}(D_{12} + D_{21})\,E_2 + \frac{1}{2}\,(D_{13} + D_{31})\,E_3$. If, therefore, the induction is derivable from a potential, $D_{12} = \frac{1}{2}\,(D_{12} + D_{21})$ or $D_{12} = D_{21}$. Similarly $D_{13} = D_{31}$ and $D_{23} = D_{32}$. By a proper choice of axes the remaining six D's can be reduced to three. In the case of isotropic dielectrics $D_{11} = D_{22} = D_{33}$ and $4\pi D_{11}$ corresponds to k, the dielectric constant.

SECTION 3

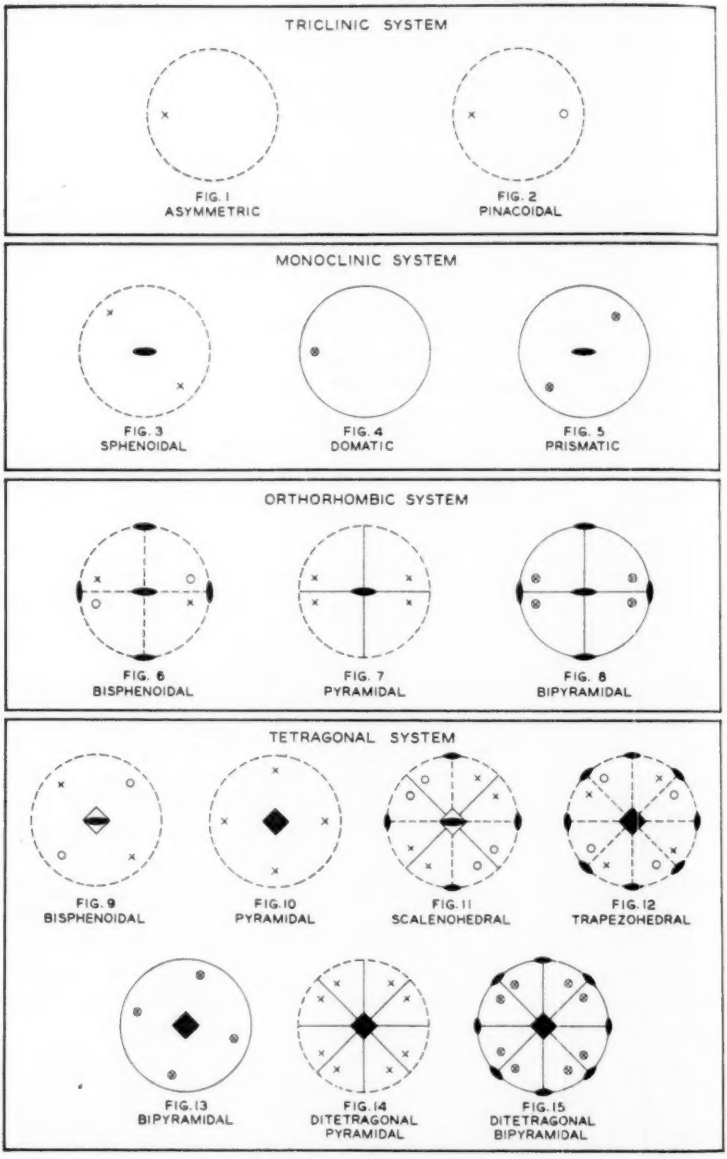
The Symmetry of Crystals

If a crystal has certain sorts of symmetry the number of constants required to describe each property is materially reduced. For this reason we now turn our attention to a study of symmetry.

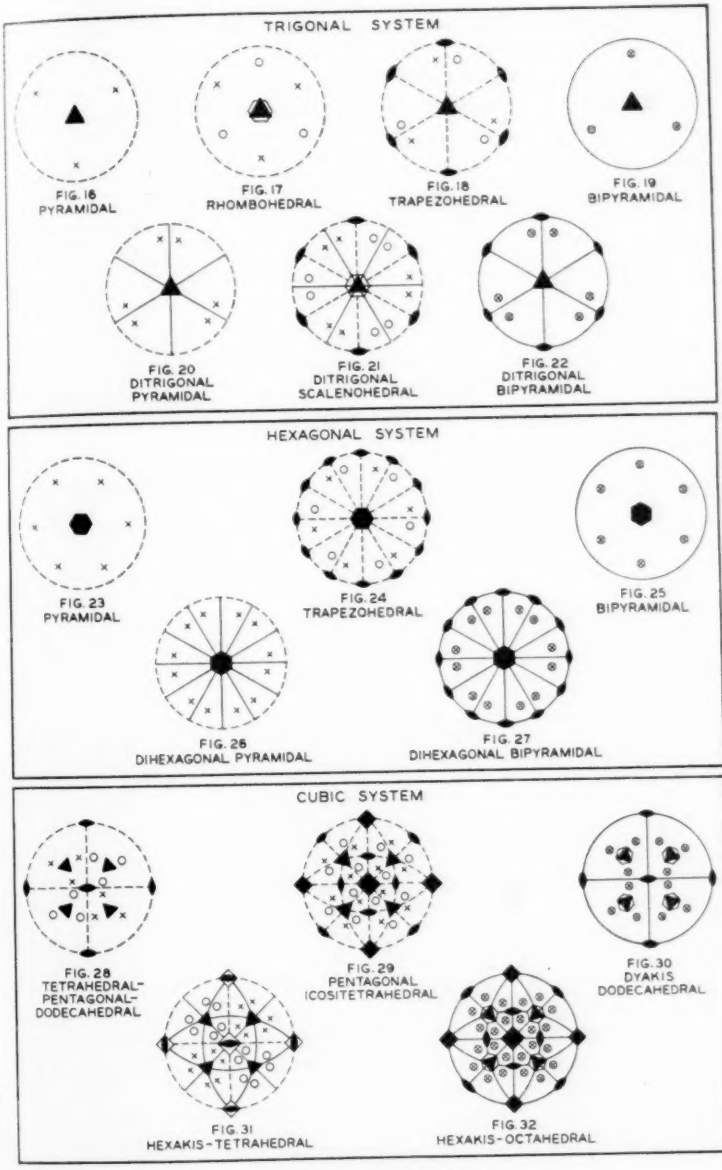
In general, plotting a vector property of the medium for a crystal gives a complicated surface which we shall call a property surface. Each property surface of a homogeneous isotropic medium is a sphere.

Because of the orderly arrangement of matter in a crystal, the property surfaces of crystalline media are commonly symmetrical. If a casting of a property surface were made it might fit into its mold in several positions. A property surface for quartz for example, if lifted from its mold and rotated through a third of a turn about the proper axis, would fit back into the mold. That is, quartz has a three fold axis. The natural requirement that molecules be laid down in a way economical of space limits the kinds of symmetry possible for crystals to axes of two fold (binary) symmetry, of three fold (trigonal), of four fold and of six fold symmetry, planes of reflection symmetry and combinations of axis-reflection symmetry, besides a simple symmetry through a center. From these elements it is possible to divide all possible property surfaces into 32 classes. No other classes built from these elements could be self-consistent.

A diagram study will prove this point. On a sphere let us mark axes of two fold symmetry by means of a solid boat shaped figure, three fold with a solid triangle, four fold with a square, six fold with a hexagon, planes of symmetry with a solid line (great circle) and combination axis reflection, by means of similar hollow figures. Finally, we shall project the sphere



Figs. 1–15—Crystal classes.



Figs. 16–32—Crystal classes.

and markings onto a plane through the center. Figures 1 to 32 is a set of such diagrams. Fig. 23 for instance shows a six fold axis. Fig. 1 represents a medium with no symmetry whatever. The cross represents a typical vector property, the vector piercing the sphere above the projection sheet. If the vector pierced below the sheet it would be marked with a circle. The dashed circle of Fig. 23 indicates the boundary of the sphere without implying it to be a plane of symmetry. The presence of six fold symmetry requires the typical vector to be shown in six places. If an axis of two fold symmetry is added at right angles to the six fold axis, it must appear six times and the typical vector must now appear twelve times, six times above and six times below the projection sheet. Continuing in this way we shall find the self-consistent classes of symmetry to be the 32 shown in the diagrams. Often the symmetry of a crystal class is expressed by means of a formula. A center of symmetry is symbolized by the letter C, a binary axis by A_2, a trigonal axis by A_3, a ternary axis by A_4, a six fold axis by A_6, a plane of reflection by P, and a combination rotation reflection by the combination symbols $\not{A}_4$ or $\not{A}_6$. In this way the symmetry formula of quartz for example, is $3A_2 \cdot A_3$.

SECTION 4

Matrix Algebra

In the solution of problems of crystal physics we are involved in the handling of many sets of linear simultaneous equations. As the matrix algebra lessens the work involved in handling sets of linear simultaneous equations we turn now to a study of matrix algebra.

Several independent variables $x_1, x_2 \ldots x_n$ are linearly related to several other independent variables $y_1, y_2 \ldots y_m$ as

$$y_1 = a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n$$
$$y_2 = a_{21}x_1 + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_m = a_{m1}x_1 + \ldots \qquad a_{mn}x_n$$

or briefly

$$y_l = \sum_{j=1}^{n} a_{lj}x_j \qquad l = 1, 2 \cdots m \ldots\ldots\ldots\ldots (4.1)$$

In most all such equations as (4.1) the variable to be summed over appears twice in the subscripts of one side. As a convention we agree to omit the summation sign and sum wherever subscripts are repeated.

Thus: $y_i = a_{ij}x_j$ is to be summed over j

again, if $x_j = b_{jk}z_k$ the z's being a third set of variables we have:

$y_i = a_{ij}b_{jk}z_k$ to be summed over j and k.

We can think of this as a special multiplication of hyperquantities a, b and z. If we define

$$c_{ik} = a_{ij}b_{jk} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.2)$$

we may go from the y's to the z's directly thru $y_i = c_{ik}z_k$. We can now consider the "table"

$$\begin{pmatrix} a_{11}a_{12} \ldots a_n \\ a_{21} \ldots \\ a_{ml} \ldots a_{mn} \end{pmatrix}$$

as being the quantity a, and the table

$$\begin{pmatrix} b_{11}b_{12} \ldots\ldots b_m \\ \ldots\ldots\ldots \\ b_{pl} \quad \ldots\ldots b_{pm} \end{pmatrix}$$

as the quantity b.

These "tables" are called matrices.

Going to eq. (4.2) we see that the quantity c is to be a "table," the typical element c_{ij} of which is to be gotten by multiplying the ith row of a by the jth column of b, term by term thus:

$$c_{ij} = a_{i1}b_{ij} + a_{i2}b_{2j} + \ldots$$

After a little practice it becomes almost automatic to form the ijth term of the product of two matrices by letting the index finger of the left hand follow across the ith row of the left matrix while the right index finger follows down the jth column of the right matrix. The fingers step along in synchronism and at each pause the quantities under the two fingers are multiplied and the product added algebraically to the accumulated sum.

The algebra of these special multiplications is not commutable, i.e. $ab \neq ba$.

Eq. (4.1) can be considered as a special case of eq. (4.2), in which the matrices x and y have one column only. In this manner a vector with components x_1 x_2 x_3 can be considered as the matrix $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

If eq. (4.1) has the same number of x's as y's we may solve (by means of determinants) for the x's in terms of the y's. We would then get a new set of equations

$$x_i = a_{ij}^{-1}\, y_j.$$

The significance of the a^{-1} is that the matrix product of a and a^{-1} is a matrix with ones on the major diagonal and all other terms zero. Whenever the product of two square matrices gives such a matrix (known as the idemfactor, I) they are said to be reciprocal. Only square matrices have reciprocals. Multiplying any matrix by the idemfactor leaves the matrix unchanged. We might consider, as part of our mathematical short hand, that eq. (4.1) was solved for x by multiplying through by a^{-1}, as

$$a^{-1}y = a^{-1}ax = Ix = x.$$

We must remember that the order must not be disturbed as the quantities are not commutable, and that only square matrices have reciprocals.

The major diagonal of a square matrix is the set of terms running diagonally from the upper left to the lower right.

A symmetrical matrix has any term $M_{ij} = M_{ji}$

An anti-symmetric or skew symmetric matrix has any term $M_{ij} = -M_{ji}$ for $i \neq j$.

Rotation Theory

The matrix algebra can be used to express a vector as a function of another vector, that is to handle such relations as exist between E and P of section 2.

There is another important aspect of matrix multiplication, that of transforming a function from one set of axes to another. Let us assume that the new set of unit axes, x_1' x_2' and x_3' are merely the old ones rotated through angle ϕ about some axis A which is a unit vector passing through the origin. From Fig. 33 we see that in the expression:

$$x_1 = a_{11}x_1' + a_{21}x_2' + a_{31}x_3'$$

the a_{ij}'s are the cosines of the angles between x_1 and the three quantities x_j'. Conversely they are the cosines of the angles between the x_j''s and x_1. Consequently, if the primed unit vectors are given in terms of the unprimed ones by the three equations

$$x_i' = a_{ij}x_j$$

then the unprimed x's are given in terms of the primed ones by the expression:

$$x_j = a_{ji}x_i'$$

This reversible relationship is well depicted by the table:

	x_1	x_2	x_3
x_1'	a_{11}	a_{12}	a_{13}
x_2'	a_{21}	a_{22}	a_{23}
x_3'	a_{31}	a_{32}	a_{33}

In this direction cosine table we can "look up" the components of any unit vector in terms of the other system.

The matrix a_{ji} is merely the matrix a_{ij} with rows and columns interchanged. a_{ji} is called the conjugate of a_{ij}. We shall denote the conjugate of any matrix M by M_c.

Obviously V' is the vector sum of the 3 components (on the new system) of each of its 3 components on the old system.

$$(V)_{\text{new}} = \begin{pmatrix} a_{11}^{-1}V_1 + a_{12}^{-1}V_2 + a_{13}^{-1}V_3 \\ a^{-1}\,V + \cdots\cdots\cdots\cdots \\ \cdots \qquad \cdots \qquad \cdots \end{pmatrix}$$

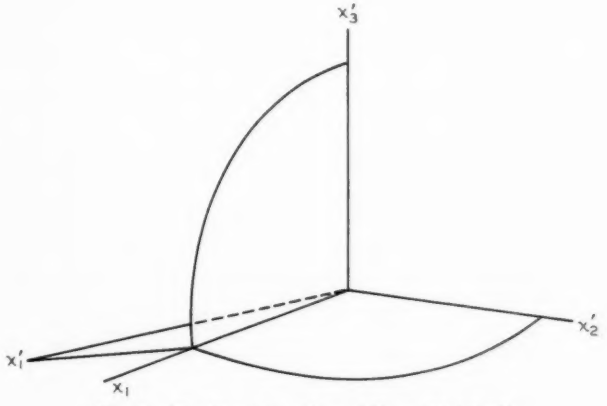


Fig. 33—The direction cosines of X_1 on $X'_1\,X'_2\,X'_3$.

If the expression giving the components of V on the new system is denoted by V' we may write

$$V' = a\,V$$

conversely

$$V = a^{-1}\,V'$$

Since x_1 is of unit length, the sum of the squares of its three components (on the primed system) is unity.

That is $\qquad a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$

similarly $\qquad a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.3)$

and $\qquad a_{31}^2 + a_{32}^2 + a_{33}^2 = 1$

Now a_c can be considered as a rotation similar to the rotation a. Consequently their product aa_c is a similar rotation. Let us consider this product.

The squares of its terms must sum to zero, row by row as in (4.3)

$$\begin{pmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{pmatrix} \begin{pmatrix} a_{11}a_{21}a_{31} \\ a_{12}a_{22}a_{32} \\ a_{13}a_{23}a_{33} \end{pmatrix} = \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \end{pmatrix}$$

Because of the relations $a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$, etc., we see that the terms of the third matrix are zero for all terms not on the major diagonal. Therefore, aa_c is an idemfactor and the reciprocal matrix of a is the same as its conjugate matrix.

$$a_c = a^{-1} \qquad (4.4)$$

Also x_1' is of unit length, and the sum of the squares of its components on the unprimed system is unity. Thus we find:

$$\begin{aligned} a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1 \\ a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1 \\ a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1 \end{aligned} \qquad (4.5)$$

We now introduce from vector analysis the concept of the scalar product. The scalar product of two vectors u and v is $u_c v$. It is the product of the lengths of the two vectors and the cosine of the angle between them.

If we take the scalar product of x_1 and x_2 as expressed in the primed system we have, since they are mutually perpendicular:

$$(a_{11}, a_{12}, a_{13}) \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0$$

Similarly multiplying x_2 and x_3 scalarly, and x_3 and x_1 we find:

$$\begin{aligned} a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} &= 0 \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} &= 0 \\ a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} &= 0 \end{aligned} \qquad (4.5)$$

If we multiply x_1' and x_2' etc. as expressed on the unprimed system we get the relations:

$$\begin{aligned} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0 \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0 \\ a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} &= 0 \end{aligned} \qquad (4.7)$$

The vector product of two vectors u and v requires the defining of a special matrix, the cross matrix.

$$u \times = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \qquad (4.8)$$

We note that this is formed by writing zeros on the major diagonal, then going back from the lower right corner writing u_1, u_2 and u_3 around the edges. We then make the lower left term negative, then operate on the opposite side of the major diagonal so as to make the matrix skew symmetric.

The reciprocal of any matrix m is

$$m^{-1}_{ij} = \frac{M_{ji}(-1)^{i+j}}{|m|} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.9)$$

where M_{ji} is the ji minor of $|m|$.

The cross matrix has no reciprocal as for it (4.9) becomes indeterminate.

Since the vector product of two vectors u and v is another vector perpendicular to both u and v and of a length $uv \sin(uv)$ we may write, on the primed system

$x_1 \times x_2 = x_3$ in the form

$$\begin{pmatrix} 0 & -a_{13} & a_{12} \\ a_{13} & 0 & -a_{11} \\ -a_{12} & a_{11} & 0 \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = \begin{pmatrix} -a_{13}a_{22} + a_{12}a_{23} \\ a_{13}a_{21} - a_{11}a_{23} \\ -a_{12}a_{21} + a_{11}a_{22} \end{pmatrix} = \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix}$$

Matrices including vectors are equal only when their corresponding terms are equal. Hence, we get the relations

$$\begin{aligned} a_{31} &= a_{12}a_{23} - a_{13}a_{22} \\ a_{32} &= a_{13}a_{21} - a_{11}a_{23} \\ a_{33} &= a_{11}a_{22} - a_{12}a_{21} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.10)$$

Similarly we get the relations:

$$\begin{aligned} a_{11} &= a_{22}a_{33} - a_{23}a_{32} \\ a_{12} &= a_{23}a_{31} - a_{21}a_{33} \\ a_{13} &= a_{21}a_{32} - a_{22}a_{31} \\ a_{21} &= a_{32}a_{13} - a_{12}a_{33} \\ a_{22} &= a_{33}a_{11} - a_{31}a_{13} \\ a_{23} &= a_{31}a_{12} - a_{32}a_{11} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.11)$$

The 21 relations between the a_{ij}'s allow us to complete the matrix given four terms.

Several Useful Matrix Relations

The del operator is the pseudo vector $\nabla = \begin{pmatrix} \partial/\partial\, x_1 \\ \partial/\partial\, x_2 \\ \partial/\partial\, x_3 \end{pmatrix} \ldots\ldots\ldots\ldots (4.12)$

It transforms upon a rotation of axes as does an ordinary vector:

$$\nabla' = a\nabla \quad (4.13)$$

$$\text{grad } u = \nabla u_c, \text{ a matrix} \quad (4.14)$$

$$\text{div } u = \nabla_c u, \text{ a scalar} \quad (4.15)$$

$$\text{curl } u = \nabla \times u, \text{ a matrix} \quad (4.16)$$

$$\text{grad radius vector} = \nabla\rho = I, \text{ the idemfactor} \quad (4.17)$$

$$(abc\ldots)^{-1} = \ldots c^{-1}b^{-1}a^{-1} \quad (4.18)$$

$$(abc\ldots)_c c = \ldots c_c b_c a_c \quad (4.19)$$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0\ldots \\ 0 & a_{22} & 0 & 0\ldots \\ 0 & 0 & a_{33} & \ldots\ldots \\ \ldots\ldots\ldots & & & \ldots\ldots \end{pmatrix}^{-1} = \begin{pmatrix} 1/a_{11} & 0 & 0 & \ldots \\ 0 & 1/a_{22} & 0 & \ldots \\ 0 & 0 & 1/a_{33} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \quad (4.20)$$

$$(\text{Scalar times matrix})^{-1} = \frac{(\text{matrix})^{-1}}{\text{scalar}} \quad (4.21)$$

SECTION 5

The Geometry of Rotations

As a first application of the matrix algebra let us compute the a matrix for a few general rotations. Although we can consider a general rotation as one of angle ϕ about the unit vector* A, it is easier to consider a general rotation as three successive rotations about coordinate axes.

A study of Fig. 34 shows that for a counterclockwise rotation ϕ about x_1, the new components of a vector V are:

$$\begin{aligned} V_1' &= V_1 \\ V_2' &= V_2 \cos\phi + V_3 \sin\phi \\ V_3' &= -V_2 \sin\phi + V_3 \cos\phi \end{aligned}$$

whence $V' = aV$ where

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \quad (5.1)$$

* A general rotation of amount ϕ about the unit axis A is given by

$$a = AA_c + (I - AA_c)\,\text{Cos}\,\phi + \text{Sin}\,\phi\, A$$

See Vector Analysis (Gibbs Wilson, Yale Press) pp. 338.

Similarly, for a counterclockwise rotation ϕ about x_2 we have

$$a = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \quad \ldots\ldots (5.2)$$

and for a counterclockwise rotation ϕ about x_3:

$$a = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \ldots\ldots (5.3)$$

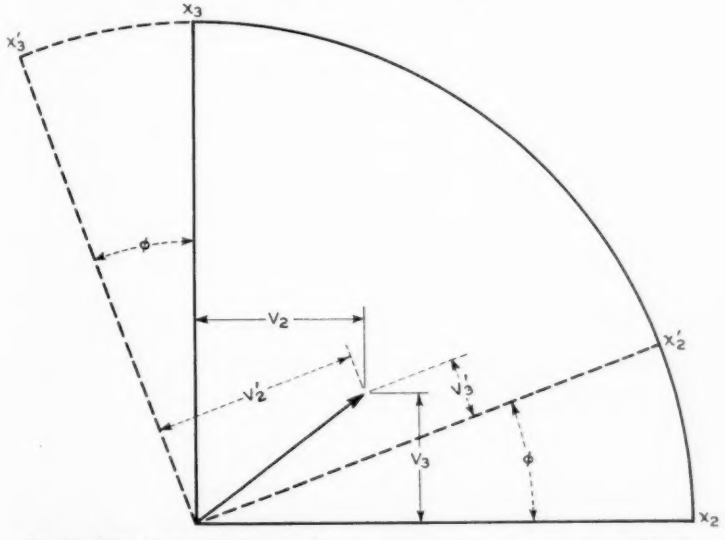


Fig. 34—The relationship between the components of a vector on one coordinate system and on another.

In the appendix we give the special transformations corresponding to the symmetry operations of the 32 crystal classes. If we have three successive rotations:

$$x' = ax$$
$$x'' = a'x'$$
$$x''' = a''x''$$

the resultant rotation is

$$x''' = a''a'ax$$

or $$x''' = Rx \quad \ldots\ldots (5.4)$$

where $$R = a''a'a \quad \ldots\ldots (5.5)$$

The I.R.E. Orientation Angles and the I.R.E. Matrix

The Institute of Radio Engineers has proposed that, for quartz crystals, all orientations be given in terms of three rotations ϕ, θ, ψ about x_3, x_2' and x_3'' respectively, starting with the plate length along x_1 width along x_2 and thickness along x_3. (Here x_3 is the z or optic axis, x_1 is the electric axis.)

Whence, here:

$$R = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and carrying out the two matrix multiplications:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & & x_2 & & x_3 & \end{array} \\ \left\{ \begin{array}{cccccc} \cos\phi\cos\theta & \cos\psi & \sin\phi\cos\theta & \cos\psi & -\sin\theta\cos\psi & x_1''' \\ -\sin\phi\sin\psi & & +\cos\phi\sin\psi & & & \\ & & & & & \\ -\cos\phi\cos\theta & \sin\psi & -\sin\phi\cos\theta & \sin\psi & & \\ -\sin\phi\cos\psi & & +\cos\phi\cos\psi & & \sin\theta\sin\psi & x_2''' \\ & & & & & \\ \cos\phi\sin\theta & & \sin\phi\sin\theta & & \cos\theta & x_3''' \end{array} \right. \end{array} \quad \ldots(5.6)$$

If we denote the unit vectors along the length, width and thickness as $P_1 P_2$ and P_3 respectively we have as a matrix defining the plate:

$$P = Rx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.7)$$

The I.R.E. orientation system is useful to the designer of crystal plates because his problem is to choose such values of ϕ, θ, ψ as to give the plate certain physical properties along its length, width and thickness. The man who cuts the plate has a different problem, that of moving the crystal (and hence the $x_1\, x_2\, x_3$ axes) about a fixed saw so that the plate cut parallel to the saw blade is what the designer ordered.

Let us consider such a system as shown in Figs. 37, 38 and 39. In Fig. 38 the crystal stands with its optic axis along P_3, its + electric axis (for right hand quartz) along P. Since the shop man considers clockwise rotation as positive we now rotate the crystal through angle U_3 about P_3 clockwise, we then turn the crystal through angle U_2 clockwise about P_1, and finally, after cutting out a slab of required thickness, we turn it clockwise through angle U_1 about P_3 to cut its length and width.

On the plate axes P_1 the crystallographic axes $x_1\, x_2\, x_3$ are now given by

$$x = rP \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.8)$$

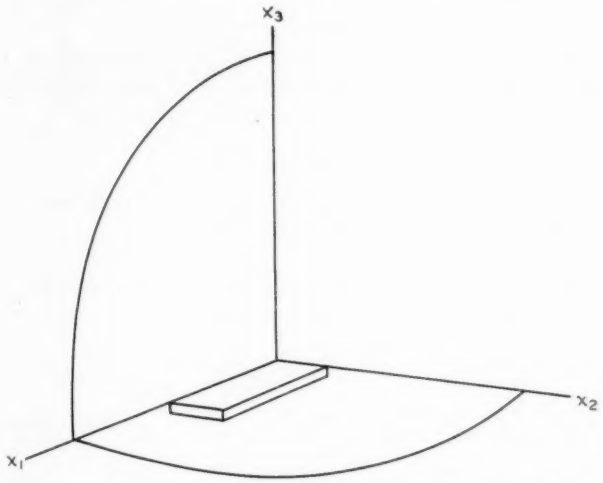


Fig. 35—The initial position (0, 0, 0) for the I.R.E. direction angles.

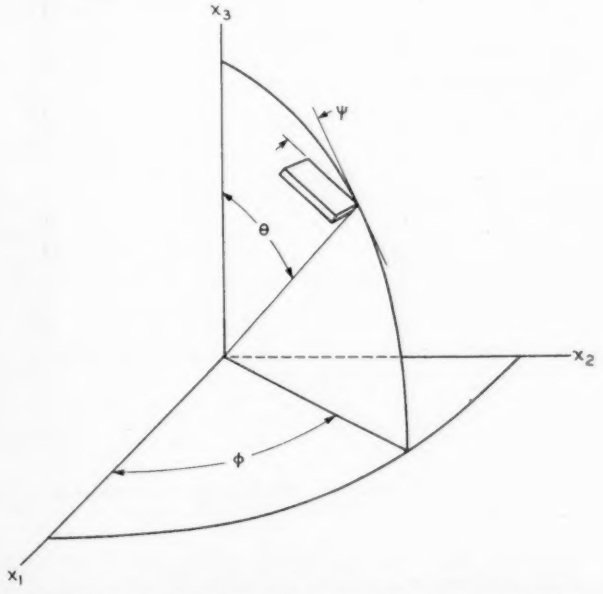


Fig. 36—The final position (Φ, Θ, Ψ) for the I.R.E. direction angles.

where

$$r = \begin{pmatrix} \cos U_1 \sin U_1 0 \\ -\sin U_1 \cos U_1 0 \\ 0 \quad 0 \quad 1 \end{pmatrix} \begin{pmatrix} 1 & \theta & \theta \\ 0 & \cos U_2 \sin U_2 \\ 0 & -\sin U_2 \cos U_2 \end{pmatrix} \begin{pmatrix} \cos U_3 \sin U_3 0 \\ -\sin U_3 \cos U_3 0 \\ 0 \quad 0 \quad 1 \end{pmatrix}$$

or

$$r = \begin{pmatrix} \cos U_1 \cos U_3 & \cos U_1 \sin U_3 & \sin U_1 \sin U_2 \\ -\sin U_1 \cos U_2 \sin U_3 & +\sin U_1 \cos U_2 \cos U_3 & \\ -\sin U_1 \cos U_3 & -\sin U_1 \sin U_3 & \cos U_1 \sin U_2 \\ -\cos U_1 \cos U_2 \sin U_3 & +\cos U_1 \cos U_2 \cos U_3 & \\ \sin U_2 \sin U_3 & -\sin U_2 \cos U_3 & \cos U_2 \end{pmatrix} \quad (5.9)$$

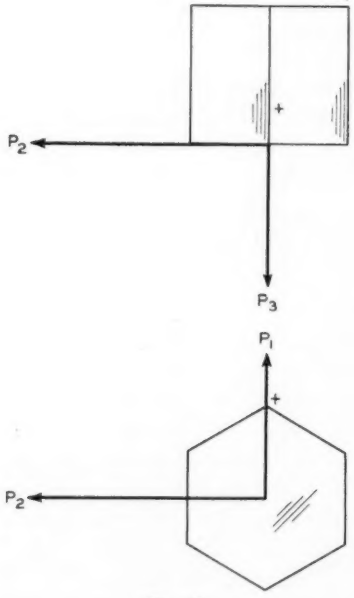


Fig. 37

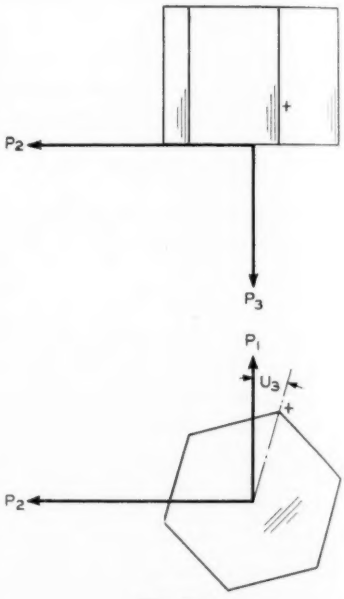


Fig. 38

Fig. 37—The (0, 0, 0) position of a shop system of direction angles.
Fig. 38—The second position of a shop system.

From (5.8) we see that $P = rx$ and hence, if this is to be the same plate the designer specified by $P = Rx$ we must have $R = r$ whence we may equate the terms of (5.6) and (5.9) to get the relations

$$\begin{aligned} \cos U_2 &= \cos\theta \text{ or } U_2 = \pm\theta \\ \tan U_1 &= \cot\psi \\ \tan U_3 &= \cot\phi \end{aligned} \qquad (5.10)$$

or

$$\begin{aligned} U_1 &= \psi - 90 + n\pi \\ U_2 &= \theta \\ U_3 &= \phi - 90 + n\pi \end{aligned}$$

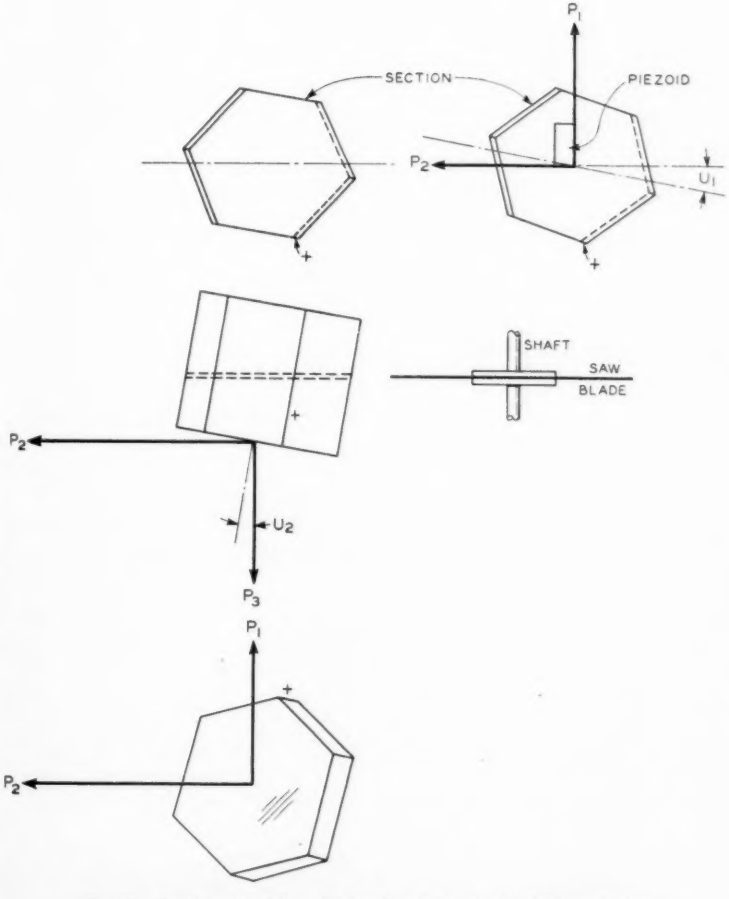


Fig. 39—Cutting the slab and trimming it to the piezoid boundaries.

Where n is any integer positive or negative, including zero. If we take

$$\begin{aligned} U_1 &= \psi - 90 \\ U_2 &= \theta \\ U_3 &= \phi + 90 \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.11)$$

The matrices are consistent term by term.

SECTION 6

Crystalline Dielectrics

As a first application of the matrix algebra considered as a linear vector function let us reconsider the problem of the crystal in an electric field.

The relations of chapter II, equation (1) can be written in the abbreviated form:

$$D = D_{ij}E \qquad \text{where} \qquad D_{rs} = D_{sr}$$

in accordance with the system of abbreviations adopted in the appendix.

If we put

$$4\pi D_{rs} = k_{rs}$$

equation (1) can be written

$$D = \frac{1}{4\pi} k_{rs} E \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.1)$$

In order to investigate the effects of crystal symmetry in determining the least number of dielectric constants that are required for a given class of symmetry it is desirable to find the electric induction D for any system of axes. Suppose that we choose a system for axes x_1', x_2', x_3' related to x_1, x_2, x_3 through the relations:

$$\begin{aligned} x_1 &= a_{11}x_1' + a_{12}x_2' + a_{13}x_3' \\ x_2 &= a_{21}x_1' + a_{22}x_2' + a_{23}x_3' \\ x_3 &= a_{31}x_1' + a_{32}x_2' + a_{33}x_3' \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.2)$$

where a_{11} is the cosine of the angle between x_1 and x_1', a_{12} is the cosine of the angle between x_1' and x_2 etc.

Equation (6.2) can be abbreviated to

$$x' = ax$$

where a is the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

It is shown in the sec. 4 that any vector $V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$ can be written on the new system of axes as V' where $V' = aV$, conversely $V = a^{-1}V'$; a^{-1} is the matrix reciprocal to a. Since the induction D and the electric field E are simple vector functions they transform as the vector V, that is:

$$D' = aD \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.3)$$

$$E' = aE \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.4)$$

But by (6.1)

$$D = \frac{1}{4\pi} kE$$

whence:

$$aD = \frac{1}{4\pi} aka_e aE$$

or

$$D' = \frac{1}{4\pi} k'E' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.5)$$

if

$$k' = aka_e \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.6)$$

We see that the form of (6.5) is the same as that of (6.1) for any set of axes if (6.6) is used to define the new dielectric matrix k.

To apply this relation (6.6) to a particular crystal let us consider a tetragonal crystal (which has its properties unchanged by a rotation of 90° about a four fold axis). Let us choose the four fold axis as x_3 and then rotate the axis 90° about x_3. In this case

$$a = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and the reciprocal matrix } a^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

whence equation (6.6) becomes:

$$k' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_{11} & k_{12} & k_{31} \\ k_{12} & k_{22} & k_{23} \\ k_{31} & k_{23} & k_{33} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} k_{22} & -k_{12} & k_{23} \\ -k_{12} & k_{11} & -k_{31} \\ k_{23} & -k_{31} & k_{33} \end{pmatrix}$$

But because of the symmetry k is unchanged by this transformation, hence

$$k = k'$$

Two matrices can be equal only if corresponding terms are equal, hence

$$k_{11} = k_{22}\,,\ k_{12} = -k_{12} = 0,\ k_{23} = -k_{31} \text{ and } k_{31} = k_{23}$$

whence

$$k_{23} = k_{31} = 0.$$

We are left then, with the dielectric constant matrix for the tetragonal bisphenoidal class:

$$k = \begin{pmatrix} k_{11} & 0 & 0 \\ 0 & k_{11} & 0 \\ 0 & 0 & k_{33} \end{pmatrix}$$

Applying other transformations possible for tetragonal crystals gives no further simplification.

If we go through all the symmetry transformations possible for the 32 classes we find that cubic crystals require but one dielectric constant, hexagonal, trigonal and tetragonal crystals require two constants, orthorhombic monoclinic and triclinic crystals require 3.

As the triclinic class has no fixed axes or planes of symmetry the reduction of its 6 constants to 3 is not so obvious. It may be seen by expanding into ordinary xyz coordinates, that $\rho_c k \rho = 1$ is the equation of an ellipsoid, (ρ is the radius vector) where the six k's are the coefficients of x^2, y^2, z^2, yz, zx and xy respectively. If we choose the coordinate axes along the axes of the ellipsoid the yz, zx and xy terms drop out and only three k's are needed. With triclinic crystals then, if we determine the axes of the ellipsoid, then choose the coordinate axes along them, only three dielectric constants are needed to completely specify the polarization in terms of the electric field. The determination of the ellipsoid axes must be made experimentally as there are no symmetry elements to guide us. It is possible to compute the positions of the axes from the 6 k's by solving a cubic equation.

The values of the k's depend on the frequency of the applied field. In crystals of low symmetry the ellipsoid axes for different frequencies do not necessarily coincide.

Another vector quantity of interest is the polarization, $P = D - \frac{1}{4\pi} E$.

Using (1) this becomes

$$P = \frac{1}{4\pi}(k - I)E \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.7)$$

SECTION 7

Quadratic Forms

Often the elements of a matrix are themselves functions of other quantities. In order to relate the elements of one matrix with those of another by means of a matrix multiplication, we may make a single column matrix of each of them. We then wish to know how a transformation of axes changes the elements of this single column matrix. Consider a symmetrical matrix b that relates two vectors u and v:

$$u = bv.$$

A transformation of axes, a, changes u and v to u' and v'. Multiplying $u = bv$ through by the prefactor a we have

$$au = abv.$$

We now replace au by its equivalent $a^{-1}v'$ whence:

$$u' = aba^{-1}v'$$

so that

$$u' = b'v'$$

if we define b' as

$$b' = aba^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7.1)$$

To be in accord with common usage we now rearrange b according to the arbitrary scheme:

$$\begin{pmatrix} b_{11} \\ b_{22} \\ b_{33} \\ b_{23} \\ b_{31} \\ b_{12} \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ . \\ . \\ B_6 \end{pmatrix}$$

We wish to know what operation to perform on B to get B' corresponding to b'. If we expand $b' = aba^{-1}$ it is easily seen that $b' = \alpha B$ where

$$\alpha = \begin{pmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{13}a_{11} & 2a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{23}a_{21} & 2a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{33}a_{31} & 2a_{31}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33} + a_{23}a_{32} & a_{21}a_{33} + a_{23}a_{31} & a_{22}a_{31} + a_{21}a_{32} \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & a_{12}a_{33} + a_{13}a_{32} & a_{13}a_{31} + a_{11}a_{33} & a_{11}a_{32} + a_{12}a_{31} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{12}a_{23} + a_{13}a_{22} & a_{13}a_{21} + a_{11}a_{23} & a_{11}a_{22} + a_{12}a_{21} \end{pmatrix} \quad . . (7.2)$$

Because we shall often need to form the α matrix from the a matrix we need an easily remembered mechanism for doing so. We notice that those are four kinds of terms in the α matrix and that the four kinds can be separated from each other by two center lines, one horizontal, one vertical. This gives us four squares of nine terms each and we can correlate each term of any square to a term of the a matrix by means of its position in the square. The terms of the upper left square are the squares of the corresponding terms of the a matrix. To form any term of the lower left square we cover the corresponding term of the a matrix with our finger and multiply the visible terms of that column. To form any term of the upper right square we cover the corresponding a term and write down *double* the product of the visible terms of that row. To form any term of the lower right square we find the corresponding a term, strike out that row and column and write down the *sum* of the remaining cross products. A study of the following diagram will help to remember these rules.

Terms are squares of corresponding a terms	Omission products doubled
Omission products	Sum of omission cross products

Fig. 40

SECTION 8

Crystal Elasticity

Stress

Consider a point P in a medium acted on by forces. If a small area is chosen about P the medium on one side of the area exerts a force on the medium on the other side. The force will depend on the size of the area and

on the direction of its normal n. We shall choose a triangular area ds such that an arbitrarily chosen set of mutually perpendicular unit axes x_1, x_2, x_3 pass through the vertices of the triangle. Let us consider the conditions of equilibrium of the tetrahedral element of volume so formed. The areas normal to x_1, x_2, x_3 are ds_1, ds_2, ds_3, respectively, and the forces per unit area acting through these faces are:

$$F_1 = \begin{pmatrix} f_{11} \\ f_{12} \\ f_{13} \end{pmatrix}, \qquad F_2 = \begin{pmatrix} f_{21} \\ f_{22} \\ f_{23} \end{pmatrix}, \qquad F_3 = \begin{pmatrix} f_{31} \\ f_{32} \\ f_{33} \end{pmatrix}$$

Any body forces (such as gravity) depend on a higher order of smallness (that is on the volume rather than on the area) and hence are negligible. Whence for equilibrium:

$$Fds = F_1 ds_1 + F_2 ds_2 + F_3 ds_3$$

But

$$ds_1 = n_1 ds, \qquad ds_2 = n_2 ds \qquad \text{and} \qquad ds_3 = n_3 ds$$

where $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is the normal to the area ds. Whence we may write: $F = fn$ where f is the matrix

$$\begin{pmatrix} f_{11} f_{12} f_{13} \\ f_{21} f_{22} f_{23} \\ f_{31} f_{32} f_{33} \end{pmatrix}$$

For the body to be in rotational equilibrium the tangential forces must balance, hence $f_{12} = f_{21}$, $f_{13} = f_{31}$ and $f_{23} = f_{32}$.

Transformation of Axes

A change of axes that transforms vectors through $F' = aF$ changes $F = fn$ to $aF = afa^{-1}an$ so that if $f' = afa^{-1}$ then $F' = f'n'$.

In order to relate the stress to other quantities through a matrix we wish to convert it into a single column matrix. We put $f_{11} = X_1$, $f_{22} = X_2$, $f_{33} = X_3$, $f_{23} = f_{32} = X_4$, $f_{31} = f_{13} = X_5$ and $f_{12} = f_{21} = X_6$.

Changing to the X representation we find

$$x' = \alpha X' \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.1)$$

where α is the matrix eq. (7.2).

Strain Theory

If the dimensions of a body change, a point $p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ is moved to $p + \sigma_p$ where $\sigma_p = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$. A neighboring point $p + u$ is moved by an amount σ_{p+u} given by $\sigma_{p+u} = (\nabla\sigma_c)_c u + \sigma_p$. The movement of $p + u$ relative to p is $\sigma = \sigma_{p+u} - \sigma_p = (\nabla\sigma_c)_c u$.

The 9 components of $(\nabla\sigma_c)_c$ describe the sort of movement in the neighborhood of a point; they are the strain coefficients. If the strain matrix is $\epsilon = (\nabla\sigma_c)_c$, a transformation $x' = ax$ causes this to become $a\epsilon a_c = (a\nabla\sigma_c a_c)_c$ and if $a\nabla = \nabla'$ and $a\sigma = \sigma'$ so that $\sigma_c a_c = \sigma_c'$ we have $\epsilon' = (\nabla'\sigma_c)_c$ if

$$\epsilon' = a\epsilon a_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.2)$$

When we arrange ϵ as a single column matrix e we shall, following custom, take $e_4 = \frac{\partial\sigma_2}{\partial x_3} + \frac{\partial\sigma_3}{\partial x_3}$, $e_5 =$ etc. This has the effect of moving the 2's of the α matrix to the conjugate position so that, while x transforms as $x' = \alpha x$, e transforms as $e' = \alpha_c^{-1} e$.

We shall take tensions as positive stress elements, and elongations as positive strain elements. The shear strain, $e_c = (0, 0, 0, 0, 0\ e_6)$ becomes upon rotating through 45° about x_3, $e_c' = \left(\frac{e_6}{2}, -\frac{e_6}{2}, 0, 0, 0, 0\right)$. This shows that to be consistent, a positive shear strain about x_3 must mean an expansion along the line $x_1 = x_2$ and an equal contraction along the line $x_1 = -x_2$.

A positive shear stress is one that tends to produce a positive shear strain.

By superposing such strain elements we see that the ϵ matrix (useful in displacement problems) may be formed from the e matrix (which is useful in stress strain relation) as

$$\epsilon = \begin{pmatrix} e_1, & \frac{1}{2}e_6 & \frac{1}{2}e_5 \\ \frac{1}{2}e_6 & e_2 & \frac{1}{2}e_4 \\ \frac{1}{2}e_5 & \frac{1}{2}e_4 & e_3 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.3)$$

This slightly awkward relation is used solely to make the "work done in straining" expressible as

$$2W = X_c e = e_c X \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.4)$$

If the e's were taken as equal to the ϵ's the work would be: $2W = X_1 e_1 + X_2 e_2 + X_3 e_3 + 2X_4 e_4 + 2X_5 e_5 + 2X_6 e_6$. This would be awkward in some later problems.

If the scalar W is to be unaffected by a transformation a we must have $W = e_c X$ unaffected. If we write

$$W = e_c \alpha^{-1} \alpha X = e_c \alpha^{-1} X'$$

we have

$$W' = W = e_c' X'$$

if

$$e_c' = e_c \alpha^{-1}$$

when

$$e' = \alpha_c^{-1} e_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.5)$$

This substantiates our previous statement.

Relation Between Stress and Strain

If the strain in an elastic body is proportioned to the stress we may write:

$$e_1 = S_{11}X_1 + S_{12}X_2 + \cdots S_{16}X_6$$
$$e_2 = S_{21}X_1 + \cdots \qquad \cdots$$

Where the S's are elastic modulii. In matrix notation:

$$e = SX \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.6)$$

Conversely $X = S^{-1}e$ or if $S^{-1} = C$

$$X = Ce \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.7)$$

The C's are called elastic constants to distinguish them from the modulii S.

As $e = SX$, $\alpha_c^{-1} e = \alpha_c^{-1} S \alpha^{-1} \alpha X$, and since $\alpha_c^{-1} e = e'$, (the representation of e on a new axis system related to the old one through the matrix a) and αX is X', then we may write $(\alpha_c^{-1} e) = (\alpha_c^{-1} S \alpha^{-1})(\alpha X)$ as:

$$e' = S'X' \quad \text{where} \quad S' = \alpha_c^{-1} S \alpha^{-1} \quad \ldots\ldots\ldots\ldots (8.8)$$

Similarly operating on $X = Ce$ we find

$$X' = C'e' \quad \text{where} \quad C' = \alpha C \alpha_c \quad \ldots\ldots\ldots\ldots\ldots (8.9)$$

The energy required to cause the strain e is

$$W = \int X_r \, de_r = \frac{1}{2} X_r e_r = \frac{1}{2} S_{rs} X_r X_s \quad \ldots\ldots\ldots (8.10)$$

whence, if W is a perfect differential,

$$S_{rs} = \frac{\partial^2 W}{\partial X_r \, \partial X_s} = \frac{\partial^2 W}{\partial X_s \, \partial X_r} = S_{sr} \quad \ldots\ldots\ldots\ldots (8.11)$$

Similarly

$$C_{rs} = C_{sr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.12)$$

This reduces the constants and modulii to 21 of each.

If a transformation is performed that is permitted by the symmetry of the medium the elastic modulus matrix is unaltered. The monoclinic system has a binary axis. If we choose this as x_3 and rotate the axes 180° about this by means of the matrix $a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have $S' = \alpha_c^{-1} S \alpha^{-1} = S$.

$$\begin{pmatrix} 1&0&0 & 0 & 0&0 \\ 0&1&0 & 0 & 0&0 \\ 0&0&1 & 0 & 0&0 \\ 0&0&0 & -1 & 0&0 \\ 0&0&0 & 0 & -1&0 \\ 0&0&0 & 0 & 0&1 \end{pmatrix} \begin{pmatrix} S_{11}S_{12}S_{13}S_{14}S_{15}S_{16} \\ S_{12}S_{22}S_{23}S_{24}S_{25}S_{26} \\ S_{13}S_{23}S_{33}S_{34}S_{35}S_{36} \\ S_{14}S_{24}S_{34}S_{44}S_{45}S_{46} \\ S_{15}S_{25}S_{35}S_{45}S_{55}S_{56} \\ S_{16}S_{26}S_{36}S_{46}S_{56}S_{66} \end{pmatrix} \begin{pmatrix} 1&0&0 & 0 & 0&0 \\ 0&1&0 & 0 & 0&0 \\ 0&0&1 & 0 & 0&0 \\ 0&0&0 & -1 & 0&0 \\ 0&0&0 & 0 & -1&0 \\ 0&0&0 & 0 & 0&1 \end{pmatrix}$$

$$\begin{pmatrix} S_{11} & S_{12} & S_{13} & -S_{14} & -S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & -S_{24} & -S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & -S_{34} & -S_{35} & S_{36} \\ -S_{14} & -S_{24} & -S_{34} & S_{44} & S_{45} & -S_{46} \\ -S_{15} & -S_{25} & -S_{35} & S_{45} & S_{55} & -S_{56} \\ S_{16} & S_{26} & S_{36} & -S_{46} & -S_{56} & S_{66} \end{pmatrix}$$

Equating terms, those whose signs differ in S and S' must vanish.

Proceeding in this way through the 32 crystal classes we arrive at the ten following matrices that cover the elastic behaviour of all 32 classes.

Triclinic System
21 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{45} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{pmatrix}$$

The C matrix is entirely analogous

$\ldots\ldots\ldots\ldots\ldots (8.13)$

Monoclinic System
x_3 axis binary
13 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{12} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{pmatrix}$$

The C matrix is entirely analogous

$\ldots\ldots\ldots\ldots\ldots (8.14)$

Phombis System
x_3 binary
9 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{23} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{pmatrix}$$

The C matrix is entirely analogous

.....(8.15)

Tetragonal System
x_3 a fourfold axis
(Classes 9, 10, 13)
7 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{11} & S_{13} & 0 & 0 & -S_{16} \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ S_{16} & -S_{16} & 0 & 0 & 0 & S_{66} \end{pmatrix}$$

The C matrix is entirely analogous

.....(8.16)

Tetragonal System
x_3 a fourfold axis
x_1 a twofold axis
(Classes 11, 12, 14, 15)
6 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{pmatrix}$$

The C matrix is entirely analogous

.....(8.17)

Trigonal System
x_3 trigonal axis
(Classes 16, 17)
7 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & -S_{15} & 0 \\ S_{12} & S_{11} & S_{13} & -S_{14} & S_{25} & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ S_{14} & -S_{14} & 0 & S_{44} & 0 & 2S_{25} \\ -S_{25} & S_{25} & 0 & 0 & S_{44} & 2S_{14} \\ 0 & 0 & 0 & 2S_{25} & 2S_{14} & 2(S_{11}-S_{12}) \end{pmatrix}$$

The C matrix is analogous except that
$C_{46} = C_{25}$
$C_{56} = C_{14}$
$C_{66} = \frac{1}{2}$
$(C_{11} - C_{12})$

.....(8.18)

Trigonal System
x_3 trigonal axis
x_1 binary (Classes 18, 20, 21)
6 modulii
(alpha quartz)

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{12} & S_{11} & S_{13} & -S_{14} & 0 & 0 \\ S_{13} & S_{13} & S_{23} & 0 & 0 & 0 \\ S_{14} & -S_{14} & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 2S_{14} \\ 0 & 0 & 0 & 0 & 2S_{14} & 2(S_{11}-S_{12}) \end{pmatrix}$$

The C matrix is analogous except that
$C_{56} = C_{14}$
$C_{66} = \frac{1}{2}$
$(C_{11} - C_{12})$

.....(8.19)

Hexagonal System
x_3 a sixfold axis
x_1 a twofold axis
(Classes 19, 22, 23, 24, 25, 26, 27)
5 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{pmatrix}$$

The C matrix is analogous except that
$C_{66} = \frac{1}{2}$
$(C_{11} - C_{12})$

.....(8.20)

Cubic System
x_1, x_2 and x_3
fourfold axes
3 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{pmatrix} \quad \text{(8.21)}$$

The C matrix is entirely analogous

Isotropic bodies
2 modulii

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & S_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & S_2 \end{pmatrix} \quad \text{(8.22)}$$

$$S_2 = 2\,(S_{11} - S_{12})$$

The C matrix is analogous except that $C_2 = \frac{1}{2}\,(C_{11} - C_{12})$

Several Elastic Ratios in common use are given here for reference:

Young's Modulus: A tension stress X divided by the component of strain in the direction of X, $Y_i = \dfrac{X_i}{e_i}$. If the coordinate axes are chosen so that the stress lies along X_1, $Y_1 = \dfrac{1}{S_{11}}$. To find the value of Y in an arbitrary direction, (θ, φ) find S' for a transformation that puts X' in the direction (θ, φ)

$$S' = \alpha_c^{-1} S \alpha^{-1}$$

Where α is taken as form (21.4). Whence we obtain:

$$\begin{aligned} \left(\frac{1}{Y_{\theta,\varphi}}\right) = {} & c_1^4 s_2^4 S_{11} + s_1^4 s_2^4 S_{22} + c_2^4 S_{33} + s_1^2 s_2^2 c_2^2 S_{44} + c_1^2 c_2^2 s_2^2 S_{55} + c_1^2 s_1^2 s_2^4 S_{66} \\ & + 2c_1^2 s_1 c_2 s_2^3 S_{56} + 2c_1 s_1^2 c_2 s_2^3 S_{46} + 2c_1 s_1 c_2^2 s_2^2 S_{56} + 2c_1 s_1^3 s_2^4 S_{26} \\ & + 2c_1^3 s_1 s_2^4 S_{16} + 2c_1^3 s_2^3 c_1 S_{15} + 2c_1^2 s_1 s_2^3 c_2 S_{14} + 2c_1^2 c_2^2 s_2^2 S_{13} \\ & + 2c_1^2 s_1^2 s_2^4 S_{12} + 2s_1^2 c_2^2 s_2^2 S_{23} + 2s_1 s_2 c_2^3 S_{34} + 2c_1 s_1 c_2^2 s_2^2 S_{45} \\ & + 2c_1 c_2^3 s_2 S_{35} + 2c_1 s_1^2 c_2 s_2^3 S_{25} + 2s_1^3 c_2 s_2^3 S_{24} \quad \text{(8.23)} \end{aligned}$$

Rigidity Modulus: The shearing stress divided by the component of shear about the axis of shearing stress. For shear about x_1,

$$N_1 = \frac{1}{S_{44}} \quad \text{(8.24)}$$

Its value in another directions can be found as $Y_{\theta\varphi}$ was above.

The bulk modulus: The change in volume per unit volume for unit hydrostatic pressure is the bulk modulus, H. For a stress $X_e = (1, 1, 1, 0, 0, 0)$

$$e = (S_{11} + S_{12} + S_{31}\,,\ S_{12} + S_{33} + S_{23}\,,\ S_{31} + S_{23} + S_{33}\,,\ \cdots)$$

$$H = (\epsilon_1 + \epsilon_2 + \epsilon_3 = S_{11} + S_{22} + S_{33} + 2S_{12} + 2S_{31} + 2S_{23}) \quad (8.25)$$

This is obviously independent of the choice of axes.

The Temperature Coefficient of the Elastic Modulii and Constants

If

$$C = C^\circ + th + t^2h^1 + t^3h^2 + \cdots \quad (8.26)^*$$

and

$$S = S^\circ + th + t^2H^1 + t^3H^2 + \cdots \quad (8.27)$$

(C° and S° denote the values of the C's and S's for some standard temperature $t = 0$) then as the transformations are

$$C' = \alpha\, C\, \alpha_e \quad \text{and} \quad S' = \alpha_e^{-1} S \alpha^{-1} \quad \text{or}$$

$$C' = \alpha\,(C^\circ + th + t^2h^1 + t^3h^{1\prime} \cdots)\alpha_e$$

and

$$S' = \alpha_e^{-1}(S^\circ + tH + t^2H^1 + t^3H^1 \cdots)\,\alpha^{-1}$$

we see that

$$C' = C^{\circ 1} + th' + t^2h^{1\prime} \cdots \quad (8.28)$$

$$S' = S^{\circ 1} + tH' + t^2H^{1\prime} \cdots \quad (8.29)$$

where

$$h' = \alpha h \alpha_e \text{ etc.} \quad (8.30)$$

$$H' = \alpha_e^{-1} H \alpha^{-1} \text{ etc.} \quad (8.31)$$

That is, the h's transform as the C's do, and the H's transform as the S's do. Consequently we may copy their respective forms from the C and S matrices for any particular crystal class.

When the temperature coefficients of the constants or modulii are known in the form:

$$C_{ij} = C^\circ_{ij}\,(1 + tT_{c_{ij}}) \quad (8.32)$$

$$S_{ij} = S'_{ij}\,(1 + tT_{s_{ij}}) \quad (8.33)$$

* The n of t^n denotes the nth power of the scalar t; the n of h^n is merely another matrix, it does not mean a power.

we may write:

$$h_{ij} = C^{\circ}_{ij} T_{c_{ij}} \quad (8.34)$$

$$H_{ij} = S^{\circ}_{ij} T_{s_{ij}} \quad (8.35)$$

Multiplying (5) by (6) we get:

$$SC = I = S^{\circ}C^{\circ} = S^{\circ}C^{\circ} + t(S^{\circ}h + HC^{\circ}) + t^2(S^{\circ}h^1 + Hh + h^1C^{\circ}) + \cdots$$

whence, for this relation to hold for all values of t:

$$S^{\circ}h + HC^{\circ} = 0 \quad (8.36)$$

whence

$$\begin{aligned} h &= -C^{\circ}HC^{\circ} \\ H &= -S^{\circ}hS^{\circ} \end{aligned} \quad (8.37)$$

also

$$S^{\circ}h^1 + Hh + H^1C^{\circ} = 0 \quad (8.38)$$

so that

$$\begin{aligned} h^1 &= hS^{\circ}h - C^{\circ}H^1C^{\circ} \\ H^1 &= HC^{\circ}H - S^{\circ}h^1S^{\circ} \end{aligned} \quad (8.39)$$

From these we can compute the h's given the H's and vice versa.

SECTION 9

Temperature Expansion

The change in the dimensions of a crystal caused by a temperature change can be considered as a strain. The shift of the terminus of a vector l relative to its origin is given from the strain matrix ϵ by the equation $\Delta l = t\epsilon l$

Since ϵ is symmetric a proper choice of axes makes it possible to make the strain per degree a diagonal matrix,

$$\Delta l = tAl \quad \text{where} \quad A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \quad (9.1)$$

As l and Δl both transform as vectors, a transformation a causes A to transform as

$$A' = aAa^{-1} \quad (9.2)$$

The elongation per unit length per degree in the direction (θ, Φ) is

$$\Delta l_{\theta\varphi} = \begin{pmatrix} \cos\theta \sin\varphi, & \sin\theta \sin\varphi, & \cos\varphi \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \begin{pmatrix} \cos\theta \sin\varphi & \cdot & \cdot \\ \sin\theta \sin\varphi & \cdot & \cdot \\ \cos = \varphi & \cdot & \cdot \end{pmatrix}$$

whence

$$\Delta l_{\theta\varphi} = A_1 \cos^2\theta \sin^2\varphi + A_2 \sin^2\theta \sin^2\varphi + A_3 \cos^2\varphi \quad \ldots\ldots (9.3)$$

The strain can easily be extended to a function of t and t^2 as follows:

$$\Delta l = tAl + t^2 Bl \quad \ldots\ldots (9.4)$$

Applying the prefactor a to both sides and putting the idemfactor in between A and l and between B and l in the form $I = a^{-1}a$ we have:

$$a\Delta l = t(aAa^{-1})al + t^2(aBa^{-1})al \quad \text{or}$$

$$\Delta l' = t\,A'l' + t^2 B'l' \quad \text{where}$$

$$A' = aAa^{-1} \qquad B' = aBa^{-1} \quad \ldots\ldots (9.5)$$

SECTION 10

Temperature Variation of the Isothermal Elastic Modulii and Stress Variation of the Temperature Expansion Coefficients

We can write the isothermal elastic modulus matrix at temperature $\theta + t$ as

$$S^i = S^{io} + tH \quad \ldots\ldots (10.1)$$

and the coefficient of temperature expansion at constant stress X as

$$\bar{A} = \bar{A}^\circ + LX \quad \ldots\ldots (10.2)$$

Let us take a unit cube of crystal about the cycle indicated in the table; starting with the cube in the unstressed unstrained state at absolute temperature θ:

Operation	Change in Stress	Change in Strain	Temp.
Heat at zero stress	0	$t\bar{A}^\circ$	θ to $\theta + t$
Apply X isothermally	X	$(S^{io} + tH)X$	$\theta + t$
Cool at Const. X	0	$-t(\bar{A}^\circ + LX)$	$\theta + t$ to θ
Apply $-X$ isothermally	$-X$	$-S^{io}X$	θ

If we sum the strain changes in this cycle to zero we have

$$H = L$$

so that we may write

$$S = S + tH \quad (10.1)$$

$$\bar{A} = \bar{A}^\circ + HX \quad (10.3)$$

This tells us that we may determine the temperature coefficients of the elastic modulii by measuring the effect of stress on the temperature expansion coefficients.

In a similar way we find that if the isothermal elastic constant matrix at temp. $\theta + t$ is:

$$C^i = C^{i\circ} + th \quad (10.4)$$

then the relation between temperature and stress at constant strain e is

$$X = tB \quad (10.5)$$

where

$$B = B^\circ + he \quad (10.6)$$

The Difference between the Specific Heats at Constant Stress and Constant Strain

Writing for the specific heats at constant stress and at constant strain σ^p and σ^v, respectively, we can perform the following cycle:

Operation	Change in Stress	Change in Strain	Temperature	Work In	Heat Out	Entropy
Heat at zero stress..	—	tA°	θ to $\theta + t$	0	$-\rho t\sigma^p$	$\frac{-\rho t\sigma^p}{\theta + t/2}$
Restore zero strain isothermally	—	$-tA^\circ$	$\theta + t$	$\frac{t^2}{2} A_e^\circ C A^\circ$	Q	$\frac{Q}{\theta + t}$
Cool at zero strain ..	—	0	$\theta + t\|\sigma\theta$	0	$\rho t\sigma^v$	$\frac{\rho t\sigma^v}{\theta + \frac{t}{2}}$

Equating the sum of the entropy changes to zero:

$$Q = \left(1 + \frac{t}{2\theta} - \frac{t^2}{4\theta^2} \cdots\right) \rho t(\sigma^p - \sigma^v)$$

Equating the work in to the heat out:

$$(\sigma^p - \sigma^v) = \frac{\theta}{\rho} \bar{A}_e^\circ C \bar{A}^\circ \quad (10.7)$$

Isothermal and Adiabatic Elastic Modulii

Let us take a unit crystal cube at temperature θ, apply any stress X adiabatically, heat it to bring the temperature back to θ at constant stress then release it isothermally. The cycle is analyzed in the table:

Operation	Change in Stress	Change in Strain	Temperature	Work In	Heat Out	Entropy Change
Apply X adiabatically	X	$S^{a}X$	θ to $\theta - t$	$\frac{1}{2}X_{c}S^{a}X$	0	0
Heat to θ at const. X	0	$t(\bar{A}^{\circ} + HX)$	$\theta - t$ to θ	$tX_{c}(A^{\circ} + HX)$	$-t\rho\sigma^{p}$	$\frac{-t\rho\sigma^{p}}{\theta - t/2}$
Remove X isothermally	$-X$	$-S^{io}X$	θ	$-\frac{1}{2}X_{c}S^{io}X$	Q	Q/θ

Summing the strains to zero:

$$(S^{io} - S^{a})X = t(\bar{A}^{\circ} + HX)$$

If we equate the total entropy change to zero we obtain an expression for Q that can be substituted in the relation "work in = Heat out." This gives us:

$$-\tfrac{1}{2}X_{c}(S^{io} - S^{a})X + tX(\bar{A}^{\circ} + HX) = \tfrac{1}{2}\frac{t^{2}\rho\sigma^{p}}{\theta}$$

and from these two expressions we derive, writing ϕ for $S^{io} - S^{a}$:

$$\phi = \frac{\theta}{\rho\sigma^{p}}(\bar{A}^{\circ} + HX)(\bar{A}^{\circ} + HX)_{c} \quad \ldots\ldots\ldots\ldots (10.8)$$

which is, to the first order of the small quantities X:

$$\phi = \frac{\theta}{\rho\sigma^{p}}\bar{A}^{\circ}\bar{A}^{\circ}_{c} + 2HX\bar{A}^{\circ}_{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10.9)$$

and since $X = C\epsilon$ we have also

$$\frac{\theta}{\rho\sigma P}(\bar{A}^{\circ} + 2HC\epsilon)\bar{A}^{\circ} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10.10)$$

Whence we see that as the stress approaches zero as a limit ϕ approaches $\phi^{\circ} = \frac{\theta}{\rho\sigma^{p}}\bar{A}^{\circ}\bar{A}^{\circ}_{c}$. If we write similarly $C^{io} - C^{a} = \psi$ we have multiplying $S^{io} = S^{a} + \phi$ by $C^{io} = C^{a} + \psi$ and dropping higher orders of small quantities:

$$\psi = -C^{io}\phi C^{io} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10.11)$$

For example we have for quartz at 20° Centigrade

$$\phi^\circ = \frac{293 \times 10^{-12}}{2.65 \times 7.37 \times 10^6} \begin{pmatrix} 14.4 \\ 14.4 \\ 7.8 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} (14.4 \;\; 14.4 \;\; 7.8 \;\; 0 \;\; -0 \;\; 0)$$

$$= \begin{pmatrix} 3.13 & 3.13 & 1.69 & 0 & 0 & 0 \\ 3.13 & 3.13 & 1.69 & 0 & 0 & 0 \\ 1.69 & 1.69 & .907 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} 10^{-15} \quad \text{and as}$$

$$S^{io} = 10^{-15} \times \begin{pmatrix} 1298 & -166 & -152 & -431 & 0 & 0 \\ -166 & 1298 & -152 & 431 & 0 & 0 \\ -152 & -152 & 990 & 0 & 0 & 0 \\ -431 & 431 & 0 & 2005 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2005 & 862 \\ 0 & 0 & 0 & 0 & 862 & 2928 \end{pmatrix}$$

whence $\phi_{11} = S^{io}_{11}\ (1 - .00241)$

$\phi_{12} = S^{io}_{12}\ (1 - .0189)$

$\phi_{13} = S^{io}_{13}\ (1 - .0111)$

$\phi_{33} = S^{io}_{33}\ (1 - .000917)$

For Rochelle Salt we have:

$$\phi = \frac{293 \times 10^{-12}}{1.79 \times 15.5 \times 10^6} \begin{pmatrix} 59.9 \\ 38.1 \\ 44.8 \\ 4 \\ 0 \\ 0 \end{pmatrix} (59.9 \;\; 38.1 \;\; 44.8 \;\; 0 \;\; 0 \;\; 0)$$

$$= \begin{pmatrix} 38.0 & 24.2 & 28.5 & 0 & 0 & 0 \\ 24.2 & 15.4 & 18.1 & 0 & 0 & 0 \\ 28.5 & 18.1 & 21.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \times 10^{-15}$$

$$\text{As } S^{io} = \begin{pmatrix} 4690 & -795 & -2180 & 0 & 0 & 0 \\ -795 & 3205 & 1691 & 0 & 0 & 0 \\ -2180 & 1691 & 2815 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6060 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3060 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8020 \end{pmatrix} \times 10^{-15}$$

So that

$$\phi_{11} = S^{io}_{11}\ (1 - .0080) \qquad \phi_{12} = S^{io}_{12}\ (1 - .0305)$$

$$\phi_{22} = S^{io}_{22}\ (1 - .0050) \qquad \phi_{13} = S^{io}_{13}\ (1 - .0103)$$

$$\phi_{33} = S^{io}_{33}\ (1 - .0076) \qquad \phi_{23} = S^{io}_{23}\ (1 - .0107)$$

$$\phi_{44} = S^{io}_{44}$$

$$\phi_{55} = S^{io}_{55}$$

$$\phi_{66} = S^{io}_{66}$$

At the temperature of maximum piezo activity the components of ϕ for Rochelle are smaller by about $3\frac{1}{2}\%$.

SECTION 11

The Piezo-electric Effect

Some crystals develop an electric charge when subjected to mechanical stresses. As far as the effect is linear it may be expressed by:

$$\begin{aligned} D_1 &= d_{11}X_1 + d_{12}X_2 \cdots d_{16}X_6 \\ D_2 &= d_{21}X_1 + \cdots \\ D_3 &= d_{31}X_1 + \cdots \qquad d_{36}X_6 \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11.1)$$

or in matrix notation

$$D = dX \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11.2)$$

where the 18 constants d_{ij} are called piezo-electric constants, and D is the electric induction.

On rotating the axes by means of a transformation a, the vector D becomes D' where $D' = aD$. The stress transforms as $X' = \alpha X$ whence $D = dX$ becomes $D' = ad\alpha^{-1}X'$ or $D' = d'X'$ where:

$$d' = ad\alpha^{-1} \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11.3)$$

If transformations permitted by symmetry are performed, the d matrix is unchanged. Class 3 has a binary axis only, if we choose this as x_3 and perform the transformation

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ we find:}$$

$$d = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$d = \begin{pmatrix} -d_{11} & -d_{12} & -d_{13} & d_{14} & d_{15} & -d_{16} \\ -d_{21} & -d_{22} & -d_{23} & d_{24} & d_{25} & -d_{26} \\ d_{31} & d_{32} & d_{33} & -d_{34} & -d_{35} & d_{36} \end{pmatrix}$$

For this to be consistent with the original d matrix the terms with conflicting signs must vanish.

Applying similar analyses to each of the 32 classes we arrive at the set of matrices:

Class 1 (asymmetric)
No symmetry

$$d = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \end{pmatrix} \quad (11.01)$$

Class 2 (triclinic pinacoidal), center of symmetry $d = 0$ ..... (11.02)

Class 3 (monoclinic sphenoidal
x_3 is binary

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & d_{15} & 0 \\ 0 & 0 & 0 & d_{24} & d_{25} & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & d_{36} \end{pmatrix} \text{ (sucrose)} \quad (11.03)$$

Class 4 (monoclinic domatic)
x_3 plane is plane of
symmetry

$$d = \begin{pmatrix} d_{11} & d_{12} & d_{13} & 0 & 0 & d_{16} \\ d_{21} & d_{22} & d_{23} & 0 & 0 & d_{26} \\ 0 & 0 & 0 & d_{34} & d_{35} & 0 \end{pmatrix} \quad (11.04)$$

Class 5 (monoclinic prismatic) center of symmetry, $d = 0$ ..... (11.05)

Class 6 (Orthorhombic
bisphenoidal)
x_1, x_2, x_3 binary

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{36} \end{pmatrix} \text{ (Rochelle)} \quad (11.06)$$

Class 7 (Orthorhombic
Pyramidal)
x_3 binary, x_1 and x_2
planes of symmetry

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{24} & 0 & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & 0 \end{pmatrix} \quad (11.07)$$

Class 8 (Orthorhombic bipyramidal), center of symmetry, $d = 0$ (11.08)

Class 9 (Tetragonal bisphenoidal) x_3 is quaternary alternating

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & d_{15} & 0 \\ 0 & 0 & 0 & -d_{15} & d_{14} & 0 \\ d_{31} & -d_{31} & 0 & 0 & 0 & d_{36} \end{pmatrix} \quad (11.09)$$

Class 10 (Tetragonal pyramidal) x_3 is quaternary

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & -d_{14} & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \quad (11.10)$$

Class 11 (Tetragonal scalenohedral) x_3 quaternary, x_1 and x_2 binary

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{36} \end{pmatrix} \quad (11.11)$$

Class 12 (Tetragonal trapezohedral) x_3 quaternary, x_1 and x_2 binary

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (11.12)$$

Class 13 (Tetragonal bipyramidal) center of symmetry, $d = 0$ (11.13)

Class 14 (Ditetragonal pyramidal) x_3 quaternary x_1 and x_2 planes of symmetry

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \quad (11.14)$$

Class 15 (Ditetragonal bipyramidal) center of symmetry, $d = 0$ (11.15)

Class 16 (Trinonal pyramidal) x_3 trigonal

$$d = \begin{pmatrix} d_{11} & -d_{11} & 0 & d_{14} & d_{15} & -2d_{22} \\ -d_{22} & d_{22} & 0 & d_{15} & -d_{14} & -2d_{11} \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \quad (11.16)$$

Class 17 (Trigonal rhombohedral) center of symmetry, $d = 0$ (11.17)

Class 18 (Trigonal trapezohedral) x_3 trigonal, x_1 binary

$$d = \begin{pmatrix} d_{11} & -d_{11} & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_{14} & -2d_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \text{(Quartz)} \\ (11.18) \end{matrix}$$

Class 19 (Trigonal bipyramidal) x_3 trigonal, x_3 plane of symmetry

$$d = \begin{pmatrix} d_{11} & -d_{11} & 0 & 0 & 0 & -2d_{22} \\ -d_{22} & d_{22} & 0 & 0 & 0 & -2d_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (11.19)$$

Class 20 (Ditrigonal pyramidal) x_3 trigonal, x_2 plane of symmetry

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} & -2d_{22} \\ -d_{22} & d_{22} & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} \text{(tourma-} \\ \text{line)} \\ (11.20) \end{matrix}$$

Class 21 (Ditrigonal scalenohedral) center of symmetry, $d = 0$ (11.21)

Class 22 (Ditrigonal bipyramidal)
x_3 trigonal, x_3 plane of symmetry
x_2 plane of symmetry

$$d = \begin{pmatrix} d_{11} & -d_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2d_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{11.22}$$

Class 23 (Hexagonal pyramidal)
x_3 Hexagonal

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & -d_{14} & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \tag{11.23}$$

Class 24 (Hexagonal trapezohedral)
x_3 hexagonal, x_1 binary

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{11.24}$$

Class 25 (Hexagonal bipyramidal) center of symmetry, $d = 0$ (11.25)

Class 26 (Dihexagonal pyramidal)
x_3 hexagonal, x_2 plane

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \tag{11.26}$$

Class 27 (Dihexagonal bipyramidal) center of symmetry, $d = 0$ (11.27)

Class 28 (Cubic tetrahedral-pentagonal-dedocahedral)
x_1, x_2, x_3 binary

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{14} \end{pmatrix} \tag{11.28}$$

Class 29 (Cubic pentagonal-icositetrahedral) $d = 0$ (11.29)

Class 30 (Cubic, dyakisdodecahedral) center of symmetry, $d = 0$ (11.30)

Class 31 (Cubic, hexakis-tetrahedral)
x_1, x_2, x_3 quaternary alternating

$$d = \begin{pmatrix} 0 & 0 & 0 & d_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{14} \end{pmatrix} \tag{11.31}$$

Class 32 (Cubic, hexakis-octahedral) center of symmetry, $d = 0$ (11.32)

Whenever a center of symmetry exists the piezo-electric property vanishes since a center of symmetry requires $d' = (-I)\, dI = -d = -d'$. Also $d = 0$ for the pentagonal icositetrahedral class.

Classes 6, 11, 12, 24, 28 and 31 polarize only by shear.

Classes 1, 3, 4, 7, 10, 14, 16, 20, 23, 26 can be polarized by hydrostatic pressure. As an example of this let us consider tourmaline (which is ditrigonal pyramidal). For hydrostatic pressure, $X_1 = X_2 = X_3$, $X_4 = X_5 = X_6 = 0$, whence from the polarization stress matrices we find, $D_1 = 0$, $D_2 = 0$, $D_3 = (2d_{31} + d_{33}) \times$ pressure. As $d_{31} = 0.75 \times 10^{-8}$ and $d_{33} = 5.8 \times 10^{-8}$ for tourmaline, we get 7.3 abcoulomlos per cm^2 per dyne per cm^2.

SECTION 12

The Converse Piezo-electric Effect

A stress X causes an electric induction

$$D = dX \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11.2)$$

and a strain

$$e = SX \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.7)$$

If the charge is allowed to leak away a further strain occurs, at constant stress. This is the strain that would be gotten if the stress were originally applied with surfaces rendered conducting:

$$e^\circ = S^\circ X \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12.1)$$

In the first sort of stress, the work per unit volume done on the crystal by establishing the stress X is:

$$W = \tfrac{1}{2}X_c e = \tfrac{1}{2}X_c SX \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.4)$$

The energy stored electrically in the medium is:

$$W_E = 2\pi D_c k^{-1} D \quad \ldots\ldots\ldots\ldots\ldots\ldots (12.2)$$

while the work done on a conducting crystal is:

$$W^\circ = \tfrac{1}{2}X_c S^\circ X \quad \ldots\ldots\ldots\ldots\ldots\ldots (12.3)$$

If a crystal be stressed in its insulated state by expenditure of energy W, the charges then absorbed by an external circuit taking up energy W_E, the strain changes from e to e° at constant stress so that the stresses perform additional work

$$W_a = X_c(e^\circ - e) = X_c(S^\circ - S)X$$

and the crystal is left containing energy W°. Whence

$$W^\circ = W - W_E + W_a \quad \ldots\ldots\ldots\ldots\ldots\ldots (12.4)$$

or:

$$\tfrac{1}{2}X_c S^\circ X = \tfrac{1}{2}X_c SX - 2\pi D_c k^{-1} D + X_c(S^\circ - S)X$$

so that:

$$X_c(S^\circ - S)X = 4\pi D_c k^{-1} D$$

If we substitute $D = dX$ we find

$$X_c(S^\circ - S)X = 4\pi X_c d_c k^{-1} dX$$

so that:

$$S^\circ - S = 4\pi d_c k^{-1} d \quad \ldots\ldots (12.5)$$

The change in strain caused by rendering the surfaces conducting is:

$$e^\circ - e = (S^\circ - S)X = 4\pi d_c k^{-1} dX \quad \ldots\ldots (12.6)$$

If the crystal be now insulated and the stress removed, an induction of opposite sign will occur and because of the assumed linear dependence of D on X the new induction will be equal to the negative of the previous one. The induction $D = -dX$ indicates an electric field:

$$E = 4\pi k^{-1} D = 4\pi k^{-1} dX \quad \ldots\ldots (12.7)$$

Also, the strain will alter by an amount $-e''$, where, since the action takes place with non-conducting surfaces:

$$e'' = SX$$

This leaves a strain on the crystal, of amount:

$$e' = e^\circ - e'' = (S^\circ - S)X \quad \ldots\ldots (12.8)$$

From (12.6), (12.7) and (12.8) it follows that:

$$e' = d_c E \quad \ldots\ldots (12.9)$$

As the medium is in just the condition that an electric field E would put the unstressed medium, (12.9) is the equation of the converse piezo-electric effect. It is to be noted that the set of constants that relates polarization and stress is the conjugate of the set that relates electric field and strain. For convenience in notation the converse effect will be written as

$$e = gE \quad \ldots\ldots (12.10)$$

where

$$g = d_c \quad \ldots\ldots (12.11)$$

Rewriting (13) as $\alpha_e^{-1} e = (\alpha_e^{-1} g a^{-1}) aE$ we see that

$$e' = g' E'$$

where

$$g' = \alpha_e^{-1} g a \quad \ldots\ldots (12.12)$$

SECTION 13

The Converse Piezo-electric Effect as a Non-Linear Function

If the strain of a crystal is not strictly a linear function of the electric field causing it we must relate the components of strain to field terms of the second power as well as to first power terms. That is, the equation $e = gE$ (which gives the strain e in terms of the electric field E through the 18 constants g) must be modified to include terms E_iE_j. All such terms are included in the symmetric matrix $(E\,E_c)$.

A transformation a that replaces E by a_cE' also replaces E_c by E'_ca so that $(E\,E_c)$ is replaced by $(a_cE'E'_ca)$, that is $(E\,E_c)$ being self-conjugate, transforms similarly to the stress matrix. We may rearrange this as a one column matrix similar to the stress matrix X, as follows:

$$(E\,E_c) = \begin{pmatrix} E_1^2 \\ E_2^2 \\ E_3^2 \\ E_2E_3 \\ E_3E_1 \\ E_1E_2 \end{pmatrix} = \bar{E} \qquad (13.1)$$

We may now relate the strain to E and $\bar{E}$ through the two matrices g and G:

$$e = gE + G\bar{E} \qquad (13.2)$$

If transformations permitted by the symmetry of the crystal are performed, g' must equal g and G' must equal G, this allows us to simplify the matrices; g is no different than before and hence vanishes for all types having centers of symmetry (and for the pentagonal icositetrahedral class).

Rewriting (1) as $\alpha_c^{-1}e = (\alpha_c^{-1}ga^{-1})aE + \alpha_c^{-1}Ga^{-1}aE$ we see that

$$e' = g'E' + G'\bar{E}$$

where

$$g' = \alpha_c^{-1}ga$$
$$G' = \alpha_c^{-1}G\alpha^{-1} \qquad (13.3)$$

The matrix G transforms as the elastic modulii matrix does but $G_{ij} \neq G_{ji}$. Applying $G' = \alpha_cG\alpha$ we arrive at the set of matrices that follow

Triclinic (36 consts)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} & G_{15} & G_{16} \\ G_{21} & G_{22} & G_{23} & G_{24} & G_{25} & G_{26} \\ G_{31} & G_{32} & G_{33} & G_{34} & G_{35} & G_{36} \\ G_{41} & G_{42} & G_{43} & G_{44} & G_{45} & G_{46} \\ G_{51} & G_{52} & G_{53} & G_{54} & G_{55} & G_{56} \\ G_{61} & G_{62} & G_{63} & G_{64} & G_{65} & G_{66} \end{pmatrix} \tag{13.4}$$

Monoclinic (20 consts)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & 0 & 0 & G_{13} \\ G_{21} & G_{22} & G_{23} & 0 & 0 & G_{26} \\ G_{31} & G_{32} & G_{33} & 0 & 0 & G_{36} \\ 0 & 0 & 0 & G_{44} & G_{45} & 0 \\ 0 & 0 & 0 & G_{54} & G_{55} & 0 \\ G_{61} & G_{62} & G_{63} & 0 & 0 & G_{66} \end{pmatrix} \tag{13.5}$$

Orthorhombic (12 consts)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & 0 & 0 & 0 \\ G_{21} & G_{22} & G_{23} & 0 & 0 & 0 \\ G_{31} & G_{32} & G_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{66} \end{pmatrix} \tag{13.6}$$

Tetragonal Classes 9, 10, 13 (10 consts)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & 0 & 0 & G_{16} \\ G_{12} & G_{11} & G_{13} & 0 & 0 & -G_{16} \\ G_{31} & G_{31} & G_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{44} & G_{45} & 0 \\ 0 & 0 & 0 & -G_{45} & G_{44} & 0 \\ G_{61} & -G_{61} & 0 & 0 & 0 & G_{66} \end{pmatrix} \tag{13.7}$$

Tetragonal Classes 11, 12, 14, 15) (7 consts)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & 0 & 0 & 0 \\ G_{12} & G_{11} & G_{13} & 0 & 0 & 0 \\ G_{31} & G_{31} & G_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{66} \end{pmatrix} \tag{13.8}$$

Trigonal (Classes 16, 17) (10 consts)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} & -G_{25} & 0 \\ G_{12} & G_{11} & G_{13} & -G_{14} & G_{25} & 0 \\ G_{31} & G_{31} & G_{33} & 0 & 0 & 0 \\ G_{41} & -G_{41} & 0 & G_{44} & G_{45} & 2G_{52} \\ -G_{52} & G_{52} & 0 & -G_{45} & G_{44} & 2G_{41} \\ 0 & 0 & 0 & 2G_{25} & 2G_{14} & 2(G_{11}-G_{22}) \end{pmatrix} \tag{13.9}$$

Trigonal (Classes 18, 20, 21) (8 constants)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & G_{14} & 0 & 0 \\ G_{12} & G_{11} & G_{13} & -G_{14} & 0 & 0 \\ G_{31} & G_{31} & G_{33} & 0 & 0 & 0 \\ G_{41} & -G_{41} & 0 & G_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{44} & 2G_{41} \\ 0 & 0 & 0 & 0 & 2G_{14} & 2(G_{11} - G_{12}) \end{pmatrix} \tag{13.10}$$

Trigonal (Classes 19, 22) (6 constants) Also Hexagonal (Classes 23, 24, 25, 26, 27)

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} & 0 & 0 & 0 \\ G_{12} & G_{11} & G_{13} & 0 & 0 & 0 \\ G_{31} & G_{31} & G_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(G_{11} - G_{12}) \end{pmatrix} \tag{13.11}$$

Cubic (Classes 28, 29, 30, 31 and 32) (5 Constants)

$$\begin{pmatrix} G_{11} & G_{12} & G_{12} & 0 & 0 & 0 \\ G_{12} & G_{11} & G_{12} & 0 & 0 & 0 \\ G_{12} & G_{12} & G_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{44} \end{pmatrix} \tag{13.12}$$

Isotropic Bodies (2 Constants)

$$\begin{pmatrix} G_{11} & G_{12} & G_{12} & 0 & 0 & 0 \\ G_{12} & G_{11} & G_{12} & 0 & 0 & 0 \\ G_{12} & G_{12} & G_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{G} \end{pmatrix} \tag{13.13}$$

$$G = 2(G_{11} - G_{12})$$

According to this analysis, all bodies suffer a change in dimensions when subjected to an electric field. These strains resulting from a field are generally much smaller than those strains $e = gE$ present only in crystals lacking a center of symmetry. For example, quartz has a strain of about 6.5×10^{-8} *cms/cm/ab* volt. Glass in a field of 1000 practical volts per cm has a strain of about 4×10^{-12}, in a 100,000 volt field it has 4×10^{-8}. Rubber in the 1000 volt field strains by about 7×10^{-8} and in the 100,000 volt field by about 7×10^{-4}. The 1st order quartz strain in these fields would be about 2.2×10^{-7} and 2.2×10^{-5} respectively.

The Second Order Piezo-electric Effect

If the induction stress relation is not strictly linear one can assume the induction to depend also on second order terms of the stress:

$$D = dX + p\,(\overline{XX_c})$$

where $(\overline{XX_c})$ is a single column matrix formed from the 21 elements of XX_c and p is a matrix of the 63 elements $p_{11,1} \ldots p_{33,3}$.

Since X transforms as $X' = \alpha X'$, (XX_c) transforms as $X'X'_c = \alpha XX_c\alpha_c$. In the same way that α was formed from a we can form a matrix $\tilde{\alpha}$ that transforms the single column matrix $(\overline{XX_c})$ through $(\overline{XX_c})' = \tilde{\alpha}(\overline{XX_c})'$.

$$aD = ad\alpha^{-1}\,X + ap(\tilde{\alpha})^{-1}\,\tilde{\alpha}(\overline{XX_c}) \text{ or}$$

$$D' = d'X' + p'(\overline{XX_c})'$$

where

$$d' = ad\alpha^{-1} \quad \text{and} \quad p' = ap(\tilde{\alpha})^{-1}$$

The first order effect is the same as before. With the relation $p' = ap(\tilde{\alpha})^{-1}$ we could perform the operations of symmetry permitted by the 32 crystal classes and obtain the reduced matrices. However since $\tilde{\alpha}$ has 484 elements we shall limit ourselves to crystals with centers of symmetry.

As X is unchanged by an inversion through the origin, α is the idemfactor for this transformation and a is $-I$, also $(\tilde{\alpha}) = I$. Therefore $D' = -D = D$ so that D vanishes.

It is to be noted that although there is a sort of reciprocity between the first order piezo effect and the converse effect, in that the matrices for one are the conjugates of the matrices of the other, there is no such reciprocity in the second order effects: if a center of symmetry exists no polarization can be brought about by stress either as a first order effect or as a second order effect; if a center of symmetry exists an electric field can cause a strain through the second order effect but not through the first order effect.

Dielectric Constants at Constant Stress and at Constant Strain

Let us consider a unit crystal cube, initially unstressed, unstrained and in zero electric field. We write k^p and k^v for the dielectric constant matrices at constant stress and constant strain, respectively, C^E as the elastic constant matrix at constant electric field E, C^o as the same for zero field. We study a cycle consisting of a strain caused by application of an electric field E at zero stress followed by a stress applied at constant E to reduce the strain to zero and completed by conducting away the electric charges at zero strain so that the body is left in its original state. The cycle is described by the table:

Operation	Change in Stress	Change in Strain	Change in Displacement Current	Change in Field	Energy Put In
Apply E	0	$d_c E$	$\frac{1}{4\pi} k^p E$	$E\frac{1}{8\pi}$	$E_c k^p E$
Apply $-e$....	$-C^E d_c E$	$-d_c E$	$-\frac{1}{4\pi}(k^p - k^v)E$	$0\frac{1}{2}$ $\frac{1}{4\pi}$	$E_c d\, C^E d_c E -$ $E_c(k^p - k^v)E$
Apply $-E$ ...	$Cd_c E$	0	$-\frac{1}{4\pi} k^v E$	$-E - \frac{1}{8\pi}$	$E_c k^v E$

whence $C = C^E$; also

$$k^p - k^v = 4\pi\, dCd_c \qquad (13.14)$$

SECTION 14

Pyro-Electricity

If the electric polarization brought about by heating some kinds of crystals is simply a function of the uniform temperature change, that is if this polarization can be produced by taking the whole body quickly from the

uniform temperature $\underline{t}_0$ to the uniform temperature $\underline{t}_0 + \underline{t}$ the pyro-electric effect could be described by the equation:

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \underline{t} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.1)$$

where ρ is the pyro-electric matrix.

This can be approached in another way by considering the polarization as due to the uniform strain. We may hence write, since $X = Ce$: (i.e. stress matrix = elastic constant matrix times the strain matrix)

$$P = dX = dCe$$

where e is the strain brought about by the temperature change t. If $A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix}$ is the temperature expansion matrix we have:

$$P = \underline{t}\, dC \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now since d has 3 rows and the A matrix has but one column the product $dC\bar{A}$ has 3 rows and one column so that we may define ρ as $dC\bar{A}$.

As D of $D = t\rho$ transforms by $D' = aD$, so does ρ:

$$\rho' = a\rho$$

When a center of symmetry exists a permitted transformation is $a = -I$, whence $\rho = -\rho' = -\rho$ so that $\rho = 0$. No pyro-electric effect (on this theory) could exist for a crystal with a center of symmetry.

If a binary axis exists and is chosen as x_3 we have

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} -\rho_1 \\ -\rho_2 \\ \rho_3 \end{pmatrix}$$

whence for this case

$$\rho = \begin{pmatrix} 0 \\ 0 \\ \rho_3 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.2)$$

If another binary axis exists at right angles to this one we find $\rho = 0$.

This is seen seriously to limit the number of classes showing this kind of pyro-electric effect. In fact we find $\rho = 0$ for classes 2, 5, 6, 8, 9, 11, 12, 13, 15, 17, 18, 19, 21, 22, 24, 25, 27, 28, 29, 30, 31 and 32. The expression $\rho = \begin{pmatrix} 0 \\ 0 \\ \rho_3 \end{pmatrix}$ describes the pyro-electric effect in the classes, 3, 7, 10, 14, 16, 20, 23 and 26, while the expression $\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ 0 \end{pmatrix}$ describes class 4, and only class 1 is described by

$$\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.3)$$

It is to be noted then that this theory excludes many classes ordinarily described as pyro-electric, such crystals as quartz in Class 18 for example. Consequently it would seem that whether or not this effect exists we must seek elsewhere for the explanation of the effect in quartz.

The effect can easily be explained as due to non-uniform temperature, which causes stress which in turn give rise to electric phenomena in piezo active crystals. For example a suddenly chilled crystal has its outer layers in a state of tension. This would produce just the pattern of positive and negative charges that one actually observes. As to whether the first effect exists, much argument between Lord Kelvin and others seems to have left the question still uncertain.

In pyro-electric crystals we would expect to find a difference in the piezo constants measured isothermally or adiabatically. If a temperature change t causes an electric displacement $D = \rho t$ the application of an electric field E should cause a temperature change t given by a relation such as:

$$t = \varphi E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14.4)$$

Also the temperature coefficient of expansion, A_a (for a crystal with faces rendered conducting) would differ from the coefficient A_i (for a crystal with an insulated surface).

If a crystal at temperature t_0 has suddenly applied to it a field E the temperature rises to $t_0 + \varphi E$ and the crystal strains, because of the converse piezo-effect, by amount $\epsilon = g_a E$ where g_a is the adiabatic converse piezo matrix. If the field is now removed isothermally a further strain $g_i E$ takes place. If the faces are short-circuited and the temperature restored to t_0 a further strain $A_a t = A_a \varphi E$ takes place and the crystal is then in its

initial state. Equating the sum of the strains to zero we find $(g_a - g_i)$ $E = A_s \varphi E$ or

$$g_a - g_i = A_s \varphi \quad \ldots\ldots\ldots\ldots\ldots\ldots (14.5)$$

Let the initial state of a crystal be, temperature $= t_0$, stress, strain and field $= 0$. If the (electrically insulated) crystal is heated by amount t, a strain $A_i t$ is caused and also an electric displacement $D = \rho t$. There now exists an electric field $E = r\pi k^{-1}\rho t$. Let this field be discharged at constant temperature, giving a further strain of $g_i E = 4\pi g_i k^{-1}\rho t$. The crystal is now short-circuited and if the initial temperature is restored a strain $-A_s t$ follows. The crystal is now in its initial state. If we equate the sum of the strains to zero we find:

$$A_i - A = 4\pi g_i k^{-1}\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (14.6)$$

SECTION 15

The Thermo-Electric Effect in Crystals

It should be possible for an electric field to be set up by a temperature gradient. Let us assume that the vector T is the temperature gradient and is related to the vector field E through the matrix Π by means of the equation:

$$E = \Pi T \quad \text{where} \quad \Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ \Pi_{21} & \Pi_{22} & \Pi_{23} \\ \Pi_{31} & \Pi_{32} & \Pi_{33} \end{pmatrix} \quad \ldots\ldots\ldots\ldots (15.1)$$

Examination shows that Π transforms through

$$\Pi' = a\Pi a_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15.2)$$

For Class 1 the Π matrix has the 9 terms of (15.1). Class 2 has a center of symmetry. For a center of symmetry $a = -I$ but $a = -I$ causes no change in (15.2) so that class 2 has 9 constants. The thermo electro effect is not killed by the presence of a center of symmetry. The ordinary thermo-electric effect of metals is a case in point.

If x_3 is a binary axis $a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and Π reduces to

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & 0 \\ \Pi_{21} & \Pi_{22} & 0 \\ 0 & 0 & \Pi_{33} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15.3)$$

Examination shows this form to answer for classes 3 and 4 and 5.

If x_1 and x_3 are binary (15.3) reduces to

$$\Pi = \begin{pmatrix} \Pi_{11} & 0 & 0 \\ 0 & \Pi_{22} & 0 \\ 0 & 0 & \Pi_{33} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15.4)$$

which described classes 6, 7 and 8.

For x_3 quarternary alternating:

$$\Pi = \begin{pmatrix} \Pi_{11} & \Pi_{12} & 0 \\ -\Pi_{12} & \Pi_{11} & 0 \\ 0 & 0 & \Pi_{33} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15.5)$$

This is found to handle classes 9 and 10. If x_3 is quarternary alternating and x_1 is binary:

$$\Pi = \begin{pmatrix} \Pi_{11} & 0 & 0 \\ 0 & \Pi_{11} & 0 \\ 0 & 0 & \Pi_{33} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15.6)$$

which is found to cover classes 11, 12, 13, 14 and 15. For classes 16, 17, 19, 23 and 25 Π reduces to the form (15.5).

If x_3 is trigonal and x_1 is binary the matrix is (15.5) which then handles cases 18, 20, 21, 22, 24, 26 and 27.

For cubic crystals, not only are x_1x_2 and x_3 binary as for matrix (15.4) but the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an A_3, for which $a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ whence, for classes 28, 29, 30 and 31 we find the matrix:

$$\Pi = \begin{pmatrix} \Pi_{11} & 0 & 0 \\ 0 & \Pi_{11} & 0 \\ 0 & 0 & \Pi_{11} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15.6)$$

Reports of a pyro electric effect in quartz should probably be attributed to nonuniform heating exciting the piezo electric effect. Reports of a pyro electric effect in such crystals as topaz and colemanite which have a center of symmetry and hence cannot be piezo electric should probably be attributed to this thermo electric effect.

SECTION 16

The Propagation of Light in Crystalline Media

Maxwell's equations are:

$$C\nabla \times B = 4\pi j$$
$$C\nabla \times E = -B$$

when C is the velocity of light in free space, E is the (vector) electric field, j is the induction current and B is the magnetic induction. In a crystalline medium the current is given by $4\pi j = k\dot{E}$ where k is the dielectric constant (matrix), whence:

$$C\nabla \times B = k\dot{E} \quad \ldots\ldots (16.1)$$

$$C\nabla \times E = -\dot{B} \quad \ldots\ldots (16.2)$$

As the divergence of the Curl is always zero:

$$\begin{aligned} \nabla_c k\dot{E} &= \nabla_c j = 0 \\ \nabla_c B &= 0 \end{aligned} \quad \ldots\ldots (16.3)$$

applying $\dfrac{\partial}{\partial t}$ to (16.1) and substituting (16.2) in the result:

$$-C^2\nabla \times \nabla \times E = k\ddot{E} \quad \text{or}$$

$$C^2(\nabla_c\nabla - \nabla\nabla_c)\,E = k\ddot{E} \quad \ldots\ldots (16.4)$$

We shall try as a solution:

$$\dot{E} = \dot{E}_0 e^{i(qn_c r - \omega t)} \quad \ldots\ldots (16.5)$$

where E_0 is the vector amplitude of the electric field, i is $\sqrt{-1}$, r is the radius vector from the origin to any point, q is a constant, n is the unit normal (at r) of surfaces of equal phase, and ω is 2π times the frequency of E.

Substituting (16.5) in (16.4) we find:

$$\dot{E} - n\dot{E}_c n = \frac{\omega^2}{q^2 c^2} k\dot{E} \quad \ldots\ldots (16.6)$$

Examination of (16.5) shows that $\dfrac{\omega}{q}$ is the phase velocity along n. Writing $4\pi k^{-1} j$ for $\dot{E}$ and V^2 for $\dfrac{\omega^2}{q^2}$ we have:

$$k^{-1} j - n j_c k^{-1} n = \frac{V^2}{c^2} j \quad \ldots\ldots (16.7)$$

This equation is independent of the absolute value of j so let us restrict j to being a unit vector.

$$\nabla_c k\dot{E} = 0 = \nabla_c k\dot{E}_0 e^{i(qn_c r - \omega t)} = j_c n i q e^{i(qn_c r - \omega t)}$$

whence $$j_c n = 0 \quad \ldots\ldots (16.8)$$

That is, the current is always normal to the direction of propagation.

Multiplying (16.7) thru by the prefactor j_c and cancelling the term in $j_c n$ we have:

$$j_c k^{-1} j = \frac{V^2}{c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16.9)$$

This tells us that the velocity is a single valued function of the direction of the current.

With the idemfactor I, (16.7) may be written:

$$\left(k^{-1} - \frac{V^2}{c^2} I\right) j = n(k^{-1} j)_c n$$

If we multiply this thru by $\left(k^{-1} - \frac{V^2}{c^2} I\right)^{-1}$ we get

$$j = \left(k^{-1} - \frac{V^2}{c^2} I\right)^{-1} n(k^{-1} j)_c n \quad \ldots\ldots\ldots\ldots (16.10)$$

Multiplying this thru by n_c and dropping the scalar factor $(k^{-1} j)_c n$:

$$n_c \left(k^{-1} - \frac{V^2}{c^2} I\right)^{-1} n = 0 \quad \ldots\ldots\ldots\ldots (16.11)$$

If the axes are so chosen that k is a diagonal matrix (16.9) and (16.11) become:

$$\frac{V^2}{c^2} = \frac{j_1^2}{k_{11}} + \frac{j_2^2}{k_{22}} + \frac{j_3^2}{k_{33}} \quad \ldots\ldots\ldots\ldots\ldots (16.12)$$

$$\frac{n_1^2}{\frac{1}{k_{11}} - \frac{V^2}{C^2}} + \frac{n_2^2}{\frac{1}{k_{22}} - \frac{V^2}{C^2}} + \frac{n_3^3}{\frac{1}{k_{33}} - \frac{V^2}{C^2}} = 0 \quad \ldots\ldots\ldots (16.13)$$

Examination of (16.13) shows that (16.11) must have two values of V^2 for each value of the vector normal n. As V^2 is a single valued function of j there must be two distinct values of j (j' and j'' say) for any particular n; and given n, only waves having their current vectors in the directions of j' and j'' can be propagated. A ray in the direction N but not having its j in one of the directions j' or j'' will be broken up into two components having their current vectors along j' and j'' respectively.

If the velocity V_1 corresponds to j' and V_2 to j'' we have by means of (16.10), since $n' = n''$:

$$j'_c j'' = n_c \left(k^{-1} - \frac{V_1^2}{c^2} I\right)_e^{-1} \left(k^{-1} - \frac{V_2^2}{c^2} I\right)^{-1} n(k^{-1} j')_c n(k^{-1} j'')_c n$$

(The quantities in the braces are scalar)

By means of the identity

$$(u^{-1} - v^{-1}) = -u^{-1}(u - v)v^{-1}$$

since

$$k^{-1} - \frac{V_1^2}{c^2} I - \left(k^{-1} - \frac{V_2^2}{c^2} I\right) = \frac{V_2^2 - V_1^2}{c^2} I$$

the idemfactor can be multiplied into an adjacent matrix giving

$$n_c \left(k^{-1} - \frac{V_1^2}{c^2} I\right)^{-1} \left(k^{-1} - \frac{V_2^2}{c^2} I\right)^{-1} n$$

$$= \frac{c^2}{V_2^2 - V_1^2} n_c \left(k^{-1} - \frac{V_1^2}{c^2} I\right)^{-1} - \left(k^{-1} - \frac{V_2^2}{c^2} I\right)^{-1} n$$

$$= 0 - 0 = 0$$

so that j' and j'' are mutually perpendicular.

SECTION 17

The Electro-Optic Effect

The velocity of light in a crystalline medium is a single valued function of the unit current vector j

$$\frac{V^2}{c^2} = j_c k^{-1} j \quad \ldots\ldots\ldots\ldots\ldots\ldots (16.11)$$

where c is the velocity of light in vacuo and k is the dielectric matrix, also $j = \dot{D}$ where $\dot{D}$ is $\frac{dD}{dt}$.

We developed the induction as a linear function of the electric field, deriving the relation:

$$4\pi D = kE \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.1)$$

If the induction is not a linear function of the electric field we can improve on eq. (6.1) by adding second order terms:

$$4\pi D_i = k_{i1}E_1 + k_{i2}E_2 + k_{i3}E_3 + k_{i1}E_1^2 + k_{i2}E_2^2 + k_{i3}E_3^2 + \tfrac{1}{2} h_{i4}E_2E_3 + \tfrac{1}{2} h_{i4}E_3E_2 + \ldots \tfrac{1}{2} h_{i6}E_1E_2$$

or

$$4\pi D_i = (k_{i1} + h_{i1}E_1 + \tfrac{1}{2} h_{i6}E_2 + \tfrac{1}{2} h_{i5}E_3)\, E_1 + (k_{i2} + \tfrac{1}{2} h_{i6}E_1 + h_{i2}E_2 + \tfrac{1}{2} h_{i4}E_3)\, E_2 + (k_{i3} + \tfrac{1}{2} h_{i5}E_1 + \tfrac{1}{2} h_{i4}E_2 + h_{i3}E_3)\, E_3 \quad \ldots\ldots\ldots\ldots (17.1)$$

Examination of (17.1) suggests that we might consider the k's as being lineraly modified by the field. Writing k as a single column matrix:

$$K = \begin{pmatrix} k_{11} \\ k_{22} \\ k_{33} \\ \\ k_{23} \\ k_{13} \\ k_{12} \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \\ . \\ \\ . \\ . \\ K_6 \end{pmatrix}$$

we may write

$$K = K^\circ + h\,E \qquad (17.2)$$

where K° is the dielectric matrix for vanishingly small fields.

We can develop the modified reciprocal matrix in the same manner:

$$k^{-1} = \begin{pmatrix} k_{11}^{-1} & k_{12}^{-1} & k_{13}^{-1} \\ k_{12}^{-1} & k_{22}^{-1} & k_{23}^{-1} \\ k_{13}^{-1} & k_{23}^{-1} & k_{33}^{-1} \end{pmatrix} \qquad K^{-1} = \begin{pmatrix} K_1^{-1} \\ K_2^{-1} \\ . \\ . \\ . \\ K_6^{-1} \end{pmatrix} = \begin{pmatrix} k_{11}^{-1} \\ k_{22}^{-1} \\ k_{33}^{-1} \\ k_{22}^{-1} \\ k_{13}^{-1} \\ k_{12}^{-1} \end{pmatrix}$$

where

$$K^{-1} = K^{-1^\circ} + zE \qquad (17.3)$$

It is to be noted that K^{-1} is not the reciprocal of K but merely a symbol for the single column matrix formed from k^{-1} in the usual way. Taking reciprocals of both sides of $k' = \alpha k a_c$ we find $(k^{-1})' = \alpha k^{-1} a_c$. That is, k^{-1} transforms exactly as did k. Whence, K^{-1} transforms exactly as K does, i. e.

$$K^{-1'} = \alpha\, K^{-1}$$

We can rewrite (17.3) as

$$\alpha\, K^{-1} = \alpha\, K^{-1^\circ} + (\alpha\, z a_c)\, aE$$

or

$$K^{-1'} = (K^{-1^\circ})' + z'E'$$

$$z' = \alpha\, z a_c \qquad (17.4)$$

In case $\alpha^{-1} = \alpha c$ the z's transform as do the conjugates of the piezoelectric constants, d. Of the transformations permitted by the symmetry

of the 32 crystal classes only those of the trigonal and hexagonal systems fail to have $\alpha^{-1} = \alpha_c$. These 12 classes must be examined individually but the other classes may have their z matrices copied from the corresponding d_c matrices.

Applying $z' = \alpha z a_c$ for a rotation of 120° about x_3 we find for class 16

$$z = \begin{pmatrix} z_{11} & -z_{22} & z_{13} \\ -z_{11} & z_{22} & z_{13} \\ 0 & 0 & z_{33} \\ z_{41} & z_{51} & 0 \\ z_{51} & -z_{41} & 0 \\ -z_{22} & -z_{11} & 0 \end{pmatrix}$$

The remaining 11 classes may be derived from class 16 by operations for which either $\alpha^{-1} = \alpha_c$ or a center of symmetry exists. Consequently, we may form our z matrices from the d_c's in all cased *if we leave out the* 2's.

The electro-optic effect can be put in terms of the polarization instead of the field by substituting in (17.3).

$$E = 4\pi (k - I)^{-1} P \quad \text{whence}$$

$$K^{-1} = K^{-1^\circ} + \eta P \qquad (17.5)$$

where

$$\eta = 4\pi z (k - I)^{-1}$$

Conversely

$$z = \frac{1}{4\pi} \eta (k - I) \qquad (17.6)$$

The η matrices transform exactly as did the z's and hence may be formed from the d_c's but omitting the 2's.

SECTION 18

The Piezo-Optical Effect

If the dielectric constants of a crystal are changed by the application of stress, this may be represented by:

$$K^{-1} = (K^{-1})^\circ + \pi X \qquad (18.1)$$

where the 36 constants $\pi_{11} \cdots \pi_{66}$ are stress-optical constants.

We may then form k^{-1} as

$$k^{-1} = \begin{pmatrix} k_{11}^{-1} + \pi_{1i} X_i, & k_{12}^{-1} + \pi_{6i} X_i, & k^{-1} + \pi_{5i} X_i \\ k_{12}^{-1} + \pi_{6i} X_i, & k_{22}^{-1} + \pi_{2i} X_i, & k_{23}^{-1} + \pi_{4i} X_i \\ k_{13}^{-1} + \pi_{5i} X_i, & k_{23}^{-1} + \pi_{4i} X_i, & k_{33}^{-1} + \pi_{3i} X_i \end{pmatrix} \qquad (18.2)$$

As the velocity of a light ray of unit current vector j is given by

$$j_e k^{-1} j = \frac{V^2}{c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16.9)$$

We can, by (18.2) and (16.9), compute the change in the velocity caused by the stress, if we know the constants π.

Altering (18.1) to $\alpha K^{-1} = \alpha K^{-1^\circ} + \alpha\pi\alpha^{-1}\alpha X$ we see that:

$$K^{-1'} = K^{-1^\circ} + \pi' X' \quad \text{where} \quad \pi' = \alpha\pi\alpha^{-1} \quad \ldots\ldots\ldots (18.3)$$

The alteration of K^{-1} can be expressed as a function of the strain by substituting ce for X in (18.1).

$$K^{-1} = K^{-1^\circ} + \pi ce = K^{-1^\circ} + me \quad \ldots\ldots\ldots\ldots (18.4)$$

$$m = \pi c, \qquad \pi = ms \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18.5)$$

Operating in (18.4) as we did on (18.1) we find m transforms as

$$m' = \alpha m \alpha_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18.6)$$

Applying the crystal symmetry operation to these matrices shows that they reduce to the following

Triclinic system
36 constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} & \pi_{15} & \pi_{16} \\ \pi_{21} & \pi_{22} & \pi_{23} & \pi_{24} & \pi_{25} & \pi_{26} \\ \pi_{31} & \pi_{32} & \pi_{33} & \pi_{34} & \pi_{35} & \pi_{36} \\ \pi_{41} & \pi_{42} & \pi_{43} & \pi_{44} & \pi_{45} & \pi_{46} \\ \pi_{51} & \pi_{52} & \pi_{53} & \pi_{54} & \pi_{55} & \pi_{56} \\ \pi_{61} & \pi_{62} & \pi_{63} & \pi_{64} & \pi_{65} & \pi_{66} \end{pmatrix}$$

The m matrix is entirely analogous

$\ldots\ldots\ldots\ldots\ldots\ldots (18.7)$

Monoclinic system
x_3 is binary
20 constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & 0 & 0 & \pi_{16} \\ \pi_{21} & \pi_{22} & \pi_{23} & 0 & 0 & \pi_{26} \\ \pi_{31} & \pi_{32} & \pi_{33} & 0 & 0 & \pi_{36} \\ 0 & 0 & 0 & \pi_{44} & \pi_{45} & 0 \\ 0 & 0 & 0 & \pi_{54} & \pi_{55} & 0 \\ \pi_{61} & \pi_{62} & \pi_{63} & 0 & 0 & \pi_{66} \end{pmatrix}$$

The m matrix is entirely analogous

$\ldots\ldots\ldots\ldots\ldots\ldots (18.8)$

Orthorhombic
system x_3
is binary
12 constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & 0 & 0 & 0 \\ \pi_{21} & \pi_{22} & \pi_{23} & 0 & 0 & 0 \\ \pi_{31} & \pi_{32} & \pi_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_{66} \end{pmatrix}$$

The m matrix is entirely analogous (Rochelle salt)

$\ldots\ldots\ldots\ldots\ldots\ldots (18.9)$

Tetragonal system
x_3 is a four-fold axis
(Classes 9, 10 & 13)
9 Constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & 0 & 0 & \pi_{16} \\ \pi_{12} & \pi_{11} & \pi_{13} & 0 & 0 & -\pi_{16} \\ \pi_{31} & \pi_{31} & \pi_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{44} & 0 \\ \pi_{61} & -\pi_{61} & 0 & 0 & 0 & \pi_{66} \end{pmatrix}$$

The m matrix is entirely analogous

.....(18.10)

Tetragonal system
x_3 is a four-fold axis
x_1 is a binary axis
(Classes 11, 12, 14 & 15)
7 Constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & 0 & 0 & 0 \\ \pi_{12} & \pi_{11} & \pi_{13} & 0 & 0 & 0 \\ \pi_{31} & \pi_{31} & \pi_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_{66} \end{pmatrix}$$

The m matrix is entirely analogous

.....(18.11)

Trigonal system
x_3 is a trigonal axis
(Classes 16 & 17)
11 Constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} & -\pi_{25} & 0 \\ \pi_{12} & \pi_{11} & \pi_{13} & -\pi_{14} & \pi_{25} & 0 \\ \pi_{31} & \pi_{31} & \pi_{33} & 0 & 0 & 0 \\ \pi_{41} & \pi_{41} & 0 & \pi_{44} & \pi_{45} & 2\pi_{52} \\ -\pi_{52} & \pi_{52} & 0 & -\pi_{45} & \pi_{44} & 2\pi_{41} \\ 0 & 0 & 0 & \pi_{25} & \pi_{14} & (\pi_{11} - \pi_{12}) \end{pmatrix}$$

The m matrix is analogous except that

$m_{46} = m_{52}$

$m_{56} = m_{41}$

$m_{66} = \dfrac{m_{11} - m_{12}}{2}$

.....(18.12)

Trigonal system
x_3 is a trigonal axis
x_1 is a binary axis
(Classes 18, 20 & 21)
8 Constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} & 0 & 0 \\ \pi_{12} & \pi_{11} & \pi_{13} & -\pi_{14} & 0 & 0 \\ \pi_{31} & \pi_{31} & \pi_{33} & 0 & 0 & 0 \\ \pi_{41} & -\pi_{41} & 0 & \pi_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{44} & 2\pi_{41} \\ 0 & 0 & 0 & 0 & \pi_{14} & (\pi_{11} - \pi_{12}) \end{pmatrix}$$

The m matrix is analogous except that

$m_{56} = m_{41}$

$m_{66} = \dfrac{m_{11} - m_{12}}{2}$

(quartz)

.....(18.13)

Hexagonal system
x_3 is a sixfold axis
x_1 is a binary axis
(Classes 19, 22, 23, 24, 25, 26 & 27)
6 Constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} & 0 & 0 & 0 \\ \pi_{12} & \pi_{11} & \pi_{13} & 0 & 0 & 0 \\ \pi_{31} & \pi_{31} & \pi_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (\pi_{11} - \pi_{12}) \end{pmatrix}$$

The m matrix is analogous except that

$m_{66} = \dfrac{m_{11} - m_{12}}{2}$

.....(18.14)

Cubic system
3 Constants

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{12} & 0 & 0 & 0 \\ \pi_{12} & \pi_{11} & \pi_{12} & 0 & 0 & 0 \\ \pi_{12} & \pi_{12} & \pi_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_{44} \end{pmatrix}$$

The m matrix is entirely analogous

.....(18.15)

For isotropic bodices, the π matrix is formed by setting $\pi_{44} = (\pi_{11} - \pi_{12})$ in the π matrix of the cubic system; the m matrix is similarly formed by putting $m_{44} = \dfrac{m_{11} - m_{12}}{2}$.

Isotropic bodies

$$\begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{12} & 0 & 0 & 0 \\ \pi_{12} & \pi_{11} & \pi_{12} & 0 & 0 & 0 \\ \pi_{12} & \pi_{12} & \pi_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi_{11}-\pi_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi_{11}-\pi_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi_{11}-\pi_{12} \end{pmatrix}$$

The m matrix has m_{44} etc $= \dfrac{m_{11}-m_{12}}{2}$ ..... (18.16)

SECTION 19

Application of the Electro and Piezo Optical Effect

In the equations $K^{-1} = K^{-1^0} + zE$ and $K^{-1} = K^{-1^0} + mE$, etc. the K^{-1}'s are to be used in forming k^{-1} for the equation giving the velocity of the light used namely $\dfrac{V^2}{C^2} = j_c k^{-1} j$. Obviously then K^{-1^0} should be formed from the squares of the reciprocals of the refractive indices, the lower three members being zero. After applying the electric field or strain a transformation of coordinates may be necessary to rediagonalize, i.e. make $K_4^{-1'} = K_5^{-1'} = K_6^{-1'} = 0$. From the rediagonalized K^{-1} we may write the new principle refractive indices by taking the reciprocals of the square roots of $K_1^{-1'}$, $K_2^{-1'}$ and $K_3^{-1'}$. It should be noted that if $K_i^{-1'} = K_i^{-1^0} + \Delta_i$ then

$$\mu_i' + \Delta\mu_i = \mu_i' - \frac{\mu_i'^3}{2}\Delta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (19.1)$$

For a given direction of the wave normal there are two velocities, a wave splitting into two components traveling with different velocities. By definition the refractive indices, μ_a and μ_b in a given direction are the normal velocities in that direction divided into the velocity of light in free space. Whence in a path length l there are $\dfrac{l\mu_a}{\lambda}$ waves in one component and $\dfrac{l\mu_b}{\lambda}$ waves in the other, where λ is the wave-length in vacuum. Consequently if l is the thickness of the crystal along that path the two components can recombine after passing through the medium but they are out of phase by $\gamma = \dfrac{l}{\lambda}(\mu_a - \mu_b)$ whole waves so that the light which entered as plane polarized will leave elliptically or circularly polarized, except when 2γ is an integer.

The quantity $B = \mu_a - \mu_b$ is known as the birefringence.

$$\gamma = \frac{lB}{\lambda} \qquad (19.2)$$

If a phase difference of $\frac{1}{30}$ wave can be just detected, using a wave length of 6000 $\mathring{A}$ and a path length $l = 1$ cm the just detectable birefringence would be $B = \frac{\gamma\lambda}{l} = 2 \times 10^{-6}$; if the path were 10 cms the detectable B would be 0.2×10^{-6}. Obviously this detectable difference between refractive indices is much smaller than could be detected by measuring each refractive index and subtracting.

It is customary to choose the coordinate system so that looking along x_2 the very lowest refractive index is for polarization in the plane of x_1 and the very highest for polarization in the plane of x_3. That is, the x_2 axis is the axis along which light should be passed to get the greatest birefringence.

Birefringence in any Direction

If the axes are so chosen that K is diagonal and $K_3 > K_2 > K_1$ then, somewhere in the plane perpendicular to x_2 are two directions, the optic axes, along which there is a single normal velocity. These directions make* equal angles V with the x_3 axis where

$$\sin V = \pm\sqrt{\frac{K_1^{-1} - K_2^{-1}}{K_1^{-1} - K_3^{-1}}}$$

or (19.3)

$$\cos V = \pm\sqrt{\frac{K_2^{-1} - K_3^{-1}}{K_1^{-1} - K_3^{-1}}}$$

Also the two refractive indices μ_a and μ_b for a wave normal making angles g_1 and g_2 with these optic axes satisfy the equation:

$$\frac{2}{\mu_a^2} = (K_1^{-1} + K_3^{-1}) + (K_1^{-1} - K_3^{-1}) \cos (g_1 - g_2)$$

$$\frac{2}{\mu_b^2} = (K_1^{-1} + K_3^{-1}) + (K_1^{-1} - K_3^{-1}) \cos (g_1 + g_2)$$

whence

$$2\left(\frac{1}{\mu_a^2} - \frac{1}{\mu_b^2}\right) = \frac{2(\mu_b - \mu_a)(\mu_a + \mu_b)}{\mu_a^2 \mu_b^2}$$

$$= (K_1^{-1} - K_3^{-1}) \cos (g_1 - g_2) - \cos (g_1 + g_2)$$

* Theory of Optics, P. Drude, pg. 320.

as $B = \mu_b - \mu_a$ is the birefringence we have:

$$B = \frac{\mu_a^2 \mu_b^2}{\mu_a + \mu_b} (K_1^{-1} - K_3^{-1}) \sin g_1 \sin g_2 \quad \ldots\ldots\ldots (19.4)$$

By spherical trigonometry:

$$\begin{aligned} \cos g_1 &= \cos V \cos\theta + \sin V \sin\theta \cos\phi \\ \cos g_2 &= \cos V \cos\theta - \sin V \sin\theta \cos\phi \end{aligned} \quad \ldots\ldots\ldots (19.5)$$

where θ is the angle the wave normal makes with x_3 and ϕ is the angle the plane containing the normal and x_3 makes with x_1.

From (19.5) it follows that:

$\sin g_1 \sin g_2 =$

$$\sqrt{(1 - \cos^2 V \cos^2\theta - \sin^2 V \sin^2\theta \cos^2\phi)^2 - 4 \sin^2 V \cos^2 V \sin^2\theta \cos^2\theta \cos^2\phi} \quad \ldots\ldots (19.6)$$

Hence if the rediagonalized K^{-1} is

$$\begin{pmatrix} K_1^{-1^\circ} + \Delta_1 \\ K_2^{-1^\circ} + \Delta_2 \\ K_3^{-1^\circ} + \Delta_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

then

$$B = \frac{\mu_a^2 \mu_b^2}{\mu_a - \mu_b}$$

$$\sqrt{\begin{aligned} &(K_1^{-1^\circ} - K_3^{-1^\circ} + \Delta_1 - \Delta_3) - (K_2^{-1^\circ} - K_3^{-1^\circ} + \Delta_2 - \Delta_3) \cos^2\theta \\ &- (K_1^{-1^\circ} - K_2^{-1^\circ} + \Delta_1 - \Delta_2) \sin^2\theta \cos^2\phi^2 - 4(K_1^{-1^\circ} - K_2^{-1^\circ} \\ &+ \Delta_1 - \Delta_2)(K_2^{-1^\circ} - K_3^{-1^\circ} + \Delta_2 - \Delta_3) \sin^2\theta \cos^2\theta \cos^2\phi \end{aligned}} \quad \ldots (19.7)$$

$$B = B_0 + \frac{\mu^3}{2} (\Delta_1 \Delta_3) \sin g_1 \sin g_2 \quad \ldots\ldots\ldots\ldots (3')$$

For most practical purposes we may take

$$\frac{\mu_a^2 \mu_b^2}{\mu_a + \mu_b} = \frac{\mu^3}{2}$$

where μ is some intermediate value of the refractive index.*

* *Note*: It might seem that as $K_i^{-1} = K_i^{-1^0} + \Delta_i$ gives us $\mu_i = \mu_i^0 - \frac{\mu_i^3}{2}\Delta_i + \frac{3}{8}\mu_i^5\Delta_i^2 \cdots$ we could form the 3 principal birefringences directly from the μ's instead of using $(6a \cdots e)$. From the μ expressions we would get $B_3 = \mu_2 - \mu_1 - \frac{1}{2}(\mu_2^3\Delta_2 - \mu_1^3\Delta_1) + \frac{3}{8}$ which differs from $6b$ if $\mu_2 \neq \mu_1$. Equation $6b$ is correct; the one from the μ expression is an approximation.

In a few special cases (18.7) may be simplified. If (ϕ, θ) falls along an optic axis $g_1 = V - V^\circ$ and $g_2 = V + V^\circ$ whence

$$B_A = \frac{\mu^3}{2}[(\Delta_1 - \Delta_2)\cos^2 V^\circ - (\Delta_2 - \Delta_3)\sin^2 V^\circ] \quad \ldots (19.61)$$

if $\theta = 0$

$$B = \mu_2 - \mu_1 + \frac{\mu_1^2 \mu_2^2}{\mu_1 + \mu_2}(\Delta_1 - \Delta_2) \quad \ldots (19.62)$$

if $\theta = 90^\circ$

$$B = (\mu_3 - \mu_1)(1 - \cos^2 \phi \sin^2 V^\circ) + \frac{\mu^3}{2}(\Delta_1 - \Delta_3 - (\Delta_1 - \Delta_2)\cos^2 \phi) \quad \ldots (19.63)$$

if $\phi = 0$

$$B = (\mu_3 - \mu_1)(\sin^2 V^\circ - \sin^2 \theta) + \frac{\mu^3}{2}(\Delta_1 - \Delta_2 - (\Delta_1 - \Delta_3)\sin \theta) \quad \ldots (19.64)$$

if $\phi = 90^\circ$

$$B = (\mu_3 - \mu_1)(1 - \cos^2 \theta \cos^2 V^\circ) + \frac{\mu^3}{2}(\Delta_1 - \Delta_3 - (\Delta_2 - \Delta_3)\cos \theta) \quad \ldots (19.65)$$

The Electro Optics of Quartz

For quartz, in the equation $K^{-1} = K^{-1^\circ} + zE$

$$Z = \begin{pmatrix} z_{11} & 0 & 0 \\ -z_{11} & 0 & 0 \\ 0 & 0 & 0 \\ z_{41} & 0 & 0 \\ 0 & -z_{41} & 0 \\ 0 & -z_{11} & 0 \end{pmatrix}$$

If E is in practical volts*
$Z_{11} = 0.47 \times 10^{-10}$
$Z_{41} = 0.20 \times 10^{-10}$

Obviously the E_3 component produces no effect so we shall examine the effects due to the components E_1 and E_2 separately. If $E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} E_1$

$$K^{-1} = \begin{pmatrix} K_1^{-1^\circ} + z_{11}E_1 \\ K_1^{-1^\circ} - z_{11}E_1 \\ K_3^{-1^\circ} \\ z_{41}E_1 \\ 0 \\ 0 \end{pmatrix}$$ which can be diagonalized

* Computed from F. Pockels data, see his Lehrbuch der Kristall-Optik, (B. G. Tuebner).

by a small rotation about x_1 of amount $\theta = \dfrac{-z_{41}E_1}{K_1^{-1} - K_3^{-1}}$ giving

$$K^{-1} = \begin{pmatrix} K_1^{-1^\circ} + z_{11}E_1 \\ K_1^{-1^\circ} - z_{11}E_1 \\ K_3^{-1^\circ} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{matrix} \mu_1 - \dfrac{\mu_1^3}{2} \times 0.47 \times 10^{-10} E_1 \\ \mu_1 + \dfrac{\mu_1^3}{2} \times 0.47 \times 10^{-10} E_1 \\ \mu_3 \end{matrix}$$

The greatest "added birefringence" is gotten by viewing along x_3, when $\Delta B = 1.544^3 \times 0.47 \times 10^{-10} E_1$. If $E_1 = 10^4 \Delta B = 1.73 \times 10^{-6}$ a quantity detectable if the path length is about 1 cm. Viewing along x_3 (the optic axis) is complicated by the rotation of the plane of polarization in quartz. Homogeneous strains have never been found to alter this rotation, but the rotation complicates and partly masks the birefringence phenomena. If $E = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} E_2$ we find

$$K^{-1} = \begin{pmatrix} K_1^{-1^\circ} \\ K_1^{-1^\circ} \\ K_3^{-1^\circ} \\ 0 \\ -z_{41}E_2 \\ -z_{11}E_2 \end{pmatrix}$$

Rotating the coordinate axes through 45° about x_3 then applying the transformation

$$a = \begin{pmatrix} 1 & 0 & \underline{a} \\ 0 & 1 & -\underline{a} \\ -\underline{a} & \underline{a} & 1 \end{pmatrix}$$

where

$$\underline{a} = \frac{-\dfrac{1}{\sqrt{2}} z_{41} E_2}{K_1^{-1} - K_3^{-1}}$$

we find:

$$K^{-1''} = \begin{pmatrix} K_1^{-1^\circ} + z_{11}E_2 \\ K_1^{-1^\circ} - z_{11}E_2 \\ K_3^{-10} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} \mu_1 - \dfrac{\mu_1^3}{2} z_{11} E_2 \\ \mu = \mu_1 + \dfrac{\mu_1^3}{2} z_{11} E_2 \\ \mu_3 \end{matrix}$$

which is identical to the μ for field along x_1, but the final axes in this case do not coincide with the final axes for $E = E_1$, but again, the greatest added birefringence is utilized by viewing along x_3. In the second case the Nicols would be best set along x_1 and x_2, i.e., at 45° to x_1' and x_2' whereas in the first case they would be best set at 45° to x_1 and x_2.

The strain Optics of Quartz

$$K^{-1} = K^{-1^\circ} + m\epsilon,$$

$$m = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} & 0 & 0 \\ m_{12} & m_{11} & m_{13} & -m_{14} & 0 & 0 \\ m_{31} & m_{31} & m_{33} & 0 & 0 & 0 \\ m_{41} & m_{41} & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{44} & m_{41} \\ 0 & 0 & 0 & 0 & m_{14} & \dfrac{m_{11} - m_{12}}{2} \end{pmatrix}$$

where*

$$\begin{aligned} m_{11} &= .138 \\ m_{12} &= .250 \\ m_{13} &= .259 \\ m_{14} &= .029 \\ m_{31} &= .258 \\ m_{33} &= .098 \\ m_{41} &= -.042 \\ m_{44} &= -.0685 \end{aligned}$$

If the strain is a simple tension along x_1,

$$K^{-1} = \begin{pmatrix} K_1^{-1^\circ} + m_{11}\underline{\epsilon_1} \\ K_1^{-1^\circ} + m_{12}\underline{\epsilon_1} \\ K_3^{-1^\circ} + m_{41}\underline{\epsilon_1} \\ m_{41}\underline{\epsilon_1} \\ 0 \\ 0 \end{pmatrix}$$

which diagonalizes, through a small transformation to:

$$K^{-1'} = \begin{pmatrix} K_1^{-1^\circ} + m_{11}\underline{\epsilon_1} \\ K_1^{-1^\circ} + m_{12}\underline{\epsilon_1} \\ K_3^{-1^\circ} + m_{31}\underline{\epsilon_1} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

applying 18.63 or 18.64 we find the birefringence along x_1 to be:

$$B_{11} = \mu_3 - \mu_1 + \frac{\mu^3}{2}(m_{12} - m_{31})\,\underline{\epsilon_1} = .0091 - .0148\,\underline{\epsilon_1}$$

Similarly the birefringence along x_2 is $B_{12} = .0091 - .225\,\underline{\epsilon_1}$ and $B_{13} = 0 - .207\,\underline{\epsilon_1}$. With a strain of 10^{-4}, which is about a tenth of the breaking strain, B_{13} would be 20.7×10^{-6}, a quantity detectable in a thickness of one millimeter.

The values of $B_{21} \cdots B_{43}$ corresponding to birefringence along x_6 for a tension along x_2 etc., can be computed in just the same way. But $B_{51} \cdots B_{63}$ require rotations of 45° about x_3 to diagonalize, so the birefringences can be computed by setting $\theta = 45°$ in equations (19.6).

* Lehrbuch der Kristalloptik—F. Pockels.

The following table summarizes these simple strain birefringence effects, the rows indicating the strain and the columns the direction of light passage.

$B_0 - .0148\ \underline{e_1}$	$B_0 - .222\ \underline{e_1}$	$.207\ \underline{e_1}$
$B_0 + .222\ \underline{e_2}$	$B_0 - .0148\ \underline{e_2}$	$.207\ \underline{e_2}$
$B_0 - .298\ \underline{e_3}$	$B_0 - .298\ \underline{e_3}$	0
$B_0 + .0536\ \underline{e_4}$	$B_0 - .0536\ \underline{e_4}$	$-.107\ \underline{e_4}$
$(B_0 + 0$	$B_0 + 0$	$-.107\ \underline{e_5})$
$(B_0 + 0$	$B_0 + 0$	$.208\ \underline{e_6})$

(the parentheses indicate the 45° transformations).

A similar table for the electro-optic effect in quartz is

$B_0 - .87 \times 10^{-10} E_1$	$B_0 + .87 \times 10^{-10} E_1$	$1.74 \times 10^{-10} E_1$
$B_0 + 0$	$B_0 + 0$	$1.74 \times 10^{-10} E_1$
$B_0 + 0$	$B_0 + 0$	0

Since a driving voltage of $E_1 = 100$ volts may, due to the building up of oscillations, cause a periodic strain of $e = 10^{-4}$ in a quartz plate, it would seem from the foregoing that 99.99% of any birefringence change must be due to the mechanical effect.

The 18° Cut Crystal

A crystal, the thickness of which is along the electric axis, x_1, the width making an angle $\theta' \doteq 18°$ with the optic axis, x_3, can be caused to oscillate with a simple motion along its length. (If θ' is not about 18° or 72° the oscillation is not a simple extension along the length, as is shown by the node which then lies diagonally across the crystal.) On a set of axes defined by the edges of the crystal block, x_1' being in the direction of the thickness or x_1, x_3' in the direction of the width and makes an angle θ' with x_3, x_2' is the length and makes an angle θ' with x_2; on these block axes the strain is e_2'. Rotating the axes about x_1' through an angle θ' we find the strain expressed on the crystal axes to be:

$$e = \alpha^{-1} e' = \begin{pmatrix} 0 \\ \cos^2 \theta' \\ \sin^2 \theta' \\ -\sin\ \theta' \cos \theta' \\ 0 \\ 0 \end{pmatrix} \underline{e_2'}$$

whence

$$K^{-1} = K^{-1^\circ} + me \quad \text{gives us}$$

$$K^{-1} = \begin{pmatrix} K_1^{-1^\circ} + (m_{12}\cos^2\theta' + m_{13}\sin^2\theta' - m_{14}\sin\theta'\cos\theta')\underline{e_2'} \\ K_1^{-1^\circ} + (m_{11}\cos^2\theta' + m_{13}\sin^2\theta' + m_{14}\sin\theta'\cos\theta')\underline{e_2'} \\ K_3^{-1^\circ} + (m_{31}\cos^2\theta' + m_{33}\sin^2\theta')\underline{e_2'} \\ \quad - (m_{41}\cos^2\theta' + m_{44}\sin\theta'\cos\varphi')\underline{e_2'} \\ 0 \\ 0 \end{pmatrix}$$

A small transformation removes the 4th term without altering the others to the first power of small quantities.

To obtain the birefringence along the width x_3', we set $\theta = \theta'$ in equation (18.65):

$$B_{3'} = (\mu_3 - \mu_1)\sin^2\theta' + \frac{\mu^3}{2}\left\{m_{12}\cos^2\theta' - m_{11}\cos^4\theta' + (m_{13} - m_{33})\sin^4\theta' - m_{14}\sin^2\theta'\frac{1+\cos^2\theta'}{2} - m_{31}\sin^2\theta'\cos^2\theta'\right\}\underline{e_2'}$$

which, for $\theta' = 18°$ is $B_3' = .00087 + .20\,\underline{e_2'}$

For the birefringence along the length x_2' we set

$\theta = 90° + \theta'$ in (6*e*) giving:

$$B = (\mu_3 - \mu_1)\cos^2\theta' + \frac{\mu^3}{2}\{m_{12}\cos^2\theta' - m_{31}\cos^4\theta' + (m_{13} - m_{11} - m_{33})\sin^2\theta'\cos^2\theta' - m_{14}\sin\theta'\cos\theta'(1+\sin^2\theta')\}\underline{e_2'}$$

which, for $\theta' = 18°$ is $B_2 = .00824 + .049\,\underline{e_2'}$.

SECTION 20

Transverse Isotropy

A material that has identical properties in all directions normal to a given line is called transversely isotropic. Any line parallel to this line may be considered as an axis of transverse isotropy.

Dielectric Properties, Optical Properties, Thermal Expansion

With respect to these, a transversely isotropic material behaves as does a uniaxial crystal, only two constants being needed to describe each. For example, the displacement current in terms of the electric field and the dielec-

tric constant matrix is $D = \frac{1}{4\pi} kE$ where, if x_3 is the axis of transverse isotropy:

$$D = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{11} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20.1)$$

Elasticity

We must find the forms of S and C, (the elastic modulus and elastic constant matrices) that are not changed by rotations about the axis of transverse isotropy. We can simplify the work by starting with the crystal class that has hexagonal symmetry only. On applying the transformation $S' = \alpha_c S \alpha = S$ for arbitrary rotations about X_3 we find no further simplification follows. Hence the S and C matrices can be copied from those for the Hexagonal Pyramidal Class.

The Piezo-Electric Effect

Again choosing x_3 as the axis of transverse isotropy and starting with hexagonal symmetry about X_3 we find that in order to be invariant to all rotations about X_3 the matrix must simplify to:

$$d = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20.2)$$

A pitch solidified in an electric field would probably exhibit this kind of piezo electric behaviour. It might also be expected to show an electro optic effect governed by a matrix like the conjugate of the above matrix.

SECTION 21

Appendix

Transformations

A counterclockwise rotation of the axes through an angle ϕ about the x_1 axis is represented by the matrices a and α as follows (where c is written for $\cos\phi$ and s for $\sin\phi$):

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & s^2 & 2sc & 0 & 0 \\ 0 & s^2 & c^2 & -2sc & 0 & 0 \\ 0 & -sc & sc & c^2-s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{pmatrix} \quad \ldots\ldots (21.1)$$

A counterclockwise rotation about x_2 is given by:

$$a = \begin{pmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ s & 0 & c \end{pmatrix}, \quad \alpha = \begin{pmatrix} c^2 & 0 & s^2 & 0 & -2sc & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ s^2 & 0 & c^2 & 0 & 2sc & 0 \\ 0 & 0 & 0 & c & 0 & s \\ sc & 0 & -sc & 0 & c^2-s^2 & 0 \\ 0 & 0 & 0 & -s & 0 & c \end{pmatrix} \quad \ldots\ldots(21.2)$$

A counterclockwise rotation about x_3 is given by:

$$a = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2-s^2 \end{pmatrix} \quad \ldots\ldots(21.3)$$

In case one wants only the value of a tensor property in a given direction not all the elements of a and α need be used, but only a row or column. A special case is that of computing such a property in the direction (θ, ϕ) of polar coordinates. The x_1' axis is chosen in this direction; x_2' and x_3' are not determined. Writing c_1 for $\cos\theta$, c_2 for $\cos\phi$, s_1 for $\sin\theta$ and s_2 for $\sin\phi$ the required matrices are

$$\begin{pmatrix} & c_1 s_2 \cdots \\ a_c = & s_1 s_2 \cdots \\ & c_2 \cdots \end{pmatrix}, \quad \alpha^{-1} = \begin{pmatrix} c_1^2 s_2^2 & \cdots \\ s_1^2 s_2^2 & \cdots \\ c_2^2 & \cdots \\ s_1 s_2 c_2 & \cdots \\ c_1 c_2 s_2 & \cdots \\ c_1 s_1 s_2^2 & \cdots \end{pmatrix} \quad \ldots\ldots(21.4)$$

From these the (11) term can be computed for any tensor.

A few special transformations needed constantly are: A rotation of 180° about x_3:

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha^{-1} = \alpha_c \quad \ldots(21.5)$$

A rotation of 90° about x_3:

$$a = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha^{-1} = \alpha_c \quad (21.6)$$

A reflection in the plane perpendicular to x_3:

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \alpha^{-1} = \alpha_c \quad (21.7)$$

A cyclic interchange where x_2 replaces x_1, etc.:
(The line making equal angles with x_1, x_2, and x_3 is a three-fold axis)

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \alpha^{-1} = \alpha_c \quad (21.8)$$

A cyclic interchange, where $-x_3$ replaces x_1, etc.:
(The line making equal angles with x_1, x_2, and x_3 is a six-fold axis of the second sort)

$$a = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \alpha^{-1} = \alpha_c \quad (21.9)$$

A rotation of 90° about x_3 combined with a reflection in the x_3 plane:

$$a = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha^{-1} = \alpha_c \quad (21.10)$$

A rotation of 60° about x_3:

$$a = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \alpha = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad \ldots (21.11)$$

To form α^{-1} substitute $-\sqrt{3}$ for $\sqrt{3}$ in α.

A rotation of 60° about x_3 combined with a reflection in x_3:

$$a = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \alpha = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad \ldots (21.12)$$

To form α^{-1} substitute $-\sqrt{3}$ for $\sqrt{3}$ in α.

A rotation of 120° about x_3:

$$a = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad \ldots (21.13)$$

To form α^{-1} substitute $-\sqrt{3}$ for $\sqrt{3}$ in α.

Inversion through the origin (a center of symmetry)

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \alpha = \begin{matrix} 1\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1 \end{matrix} = I \quad \alpha^{-1} = \alpha_c \quad \ldots\ldots\ldots (21.14)$$

A rotation of 180° about x_1:

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \alpha = \begin{matrix} 1\,0\,0\,0 & 0 & 0 \\ 0\,1\,0\,0 & 0 & 0 \\ 0\,0\,1\,0 & 0 & 0 \\ 0\,0\,0\,1 & 0 & 0 \\ 0\,0\,0\,0 & -1 & 0 \\ 0\,0\,0\,0 & 0 & -1 \end{matrix} \quad \alpha^{-1} = \alpha_c \quad \ldots (21.15)$$

A reflection in x_1 plane:

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \alpha = \begin{matrix} 1\,0\,0\,0 & 0 & 0 \\ 0\,1\,0\,0 & 0 & 0 \\ 0\,0\,1\,0 & 0 & 0 \\ 0\,0\,0\,1 & 0 & 0 \\ 0\,0\,0\,0 & -1 & 0 \\ 0\,0\,0\,0 & 0 & -1 \end{matrix} \quad \alpha^{-1} = \alpha_c \quad \ldots (21.16)$$

A reflection in the x_2 plane:

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \alpha = \begin{matrix} 1\,0\,0 & 0\,0 & 0 \\ 0\,1\,0 & 0\,0 & 0 \\ 0\,0\,1 & 0\,0 & 0 \\ 0\,0\,0 & -1\,0 & 0 \\ 0\,0\,0 & 0\,1 & 0 \\ 0\,0\,0 & 0\,0 & -1 \end{matrix} \quad \alpha^{-1} = \alpha = \alpha_c \quad \ldots (21.17)$$

In computing the electro-optic and mechanico-optic effects we need a transformation that will restore to diagonality a matrix that has very small but symmetrical off diagonal terms. This transformation we call a small transformation. Such a transformation has its matrix differing but slightly from an idemfactor.

If $$\bar{k} = \begin{pmatrix} k_{11} + \Delta_{11} & \Delta_{12} & \Delta_{31} \\ \Delta_{12} & k_{22} + \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{23} & k_{33} + \Delta_{33} \end{pmatrix}$$

we assume that it can be rediagonalized to the matrix k' by means of the transformation: $k' = \delta \bar{k} \bar{\delta}_c$ where $\delta = \begin{pmatrix} 1 & \delta_{12} & \delta_{13} \\ \delta_{21} & 1 & \delta_{23} \\ \delta_{31} & \delta_{32} & 1 \end{pmatrix}$

Since $\delta_{11}\,\delta_{21} + \delta_{12}\,\delta_{22} + \delta_{13}\,\delta_{23} = 0$ we find that, to the first order of small quantities $\delta_{ij} = -\delta_{ji}$.

Expanding $\delta \bar{k} \bar{\delta}_c$ to the first order of small quantities and equating the non-diagonal terms to zero we find that:

$$\delta_{12} = \frac{\Delta_{12}}{k_{11} - k_{22}}; \qquad \delta_{23} = \frac{\Delta_{23}}{k_{22} - k_{33}} \quad \text{and} \quad \delta_{31} = \frac{\Delta_{31}}{k_{33} - k_{11}}$$

Therefore, to the first order of small quantities Δ_{ij}:

$$\hat{\delta} \equiv \begin{pmatrix} k_{11} + \Delta_{11} & \Delta_{12} & \Delta_{31} \\ \Delta_{12} & k_{22} + \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{23} & k_{33} + \Delta_{33} \end{pmatrix} \bar{\delta}_c = \begin{pmatrix} k_{11} + \Delta_{11} & 0 & 0 \\ 0 & k_{22} + \Delta_{22} & 0 \\ 0 & 0 & k_{33} + \Delta_{33} \end{pmatrix}$$

where $\hat{\delta} = \begin{pmatrix} 1 & \dfrac{\Delta_{12}}{k_{11} - k_{22}} & \dfrac{-\Delta_{31}}{k_{33} - k_{11}} \\ \dfrac{-\Delta_{12}}{k_{11} - k_{22}} & 1 & \dfrac{\Delta_{23}}{k_{22} - k_{33}} \\ \dfrac{\Delta_{31}}{k_{33} - k_{11}} & \dfrac{-\Delta_{23}}{k_{22} - k_{33}} & 1 \end{pmatrix}$ is the transformation $x' = x$ ..... (21.18)

The electro and piezo-optic effects of biaxial crystals can be handled by these infinitesimal transformations, but uniaxial crystals and cubic crystals may require finite rotations to re-diagonalize the k^{-1} matrix. In the case of cubic crystals we note that $\begin{pmatrix} k & 0 & 0 \\ 0 & k & \Delta \\ 0 & \Delta & k \end{pmatrix}$ may be diagonalized by a rotation of 45° about x_1, giving

$$\begin{pmatrix} k & 0 & 0 \\ 0 & k + \Delta & 0 \\ 0 & 0 & k - \Delta \end{pmatrix} \text{ ..... (21.19)}$$

$\begin{pmatrix} k & 0 & \Delta_2 \\ 0 & k & \Delta_1 \\ \Delta_2 & \Delta_1 & k \end{pmatrix}$ becomes $\begin{pmatrix} k & 0 & \sqrt{\Delta_1^2 + \Delta_2^2} \\ 0 & k & 0 \\ \sqrt{\Delta_1^2 + \Delta_1^2} & 0 & k \end{pmatrix}$ upon rotation through angle

$\tan^{-1}\frac{\Delta_1}{\Delta_2}$ about x_3 and diaonalizes by then rotating through 45° about x_2 giving:

$$k'' = \begin{pmatrix} k + \sqrt{\Delta_1^2 + \Delta_2^2} & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k - \sqrt{\Delta_1^2 + \Delta_2^2} \end{pmatrix} \quad \ldots\ldots (21.20)$$

The work can be handled with single column matrices K instead of using the square matrices k. If k is a diagonal single column matrix (i.e., the single column matrix of a diagonal matrix), than the almost diagonal matrix $K + \Delta$ is diagonalized by the transformation:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{2\Delta_5}{K_1 - K_3} & \frac{2\Delta_6}{K_1 - K_2} \\ 0 & 1 & 0 & \frac{2\Delta_4}{K_2 - K_3} & 0 & \frac{-2\Delta_6}{K_1 - K_2} \\ 0 & 0 & 1 & \frac{-2\Delta_4}{K_2 - K_3} & \frac{-2\Delta_5}{K_1 - K_3} & 0 \\ 0 & \frac{-\Delta_4}{K_2 - K_3} & \frac{\Delta_4}{K_2 - K_3} & & \frac{-\Delta_6}{K_1 - K_2} & \frac{-\Delta_5}{K_1 - K_3} \\ \frac{-\Delta_5}{K_1 - K_3} & 0 & \frac{\Delta_5}{K_1 - K_3} & \frac{\Delta_6}{K_1 - K_2} & 1 & \frac{\Delta_4}{K_1 - K_3} \\ \frac{-\Delta_6}{K_1 - K_2} & \frac{\Delta_6}{K_1 - K_2} & 0 & \frac{\Delta_5}{K_1 - K_3} & \frac{\Delta_4}{K_2 - K_3} & 1 \end{pmatrix} \begin{pmatrix} K_1 + \Delta_1 \\ K_2 + \Delta_2 \\ K_1 + \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{pmatrix} =$$

$$\begin{pmatrix} K_1 + \Delta_1 \\ K_2 + \Delta_2 \\ K_3 + \Delta_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21.21)$$

which transforms vectors as $x' = \delta x$ where

$$\delta = \begin{pmatrix} 1 & \dfrac{\Delta_6}{K_1 - K_2} & \dfrac{\Delta_5}{K_1 - K_3} \\ \dfrac{-\Delta_6}{K_1 - K_2} & 1 & \dfrac{\Delta_4}{K_2 - K_3} \\ \dfrac{-\Delta_5}{K_1 - K_3} & \dfrac{-\Delta_4}{K_2 - K_3} & 1 \end{pmatrix} \quad \ldots\ldots\ldots (21.22)$$

If K_1, K_2 and K_3 are not all different the preceding analysis falls down as some terms become infinite; a finite transformation is needed in this case. The difficulty can generally be doged by applying a 45° rotation about one of the axes. Sometimes the easiest solution is to rotate through an angle ϕ about a coordinate axis then solve for the value of ϕ that will vanish certain terms. As examples of these devices we give the following:

$$\begin{pmatrix} K_1 \\ K_1 \\ K_1 \\ \Delta_4 \\ 0 \\ 0 \end{pmatrix} \text{ rotated 45° about } x_1 \text{ becomes } \begin{pmatrix} K_1 \\ K_1 + \Delta_4 \\ K_1 - \Delta_4 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \ldots\ldots (21.23)$$

$$\begin{pmatrix} K_1 \\ K_1 \\ K_1 \\ \Delta_4 \\ \Delta_5 \\ 0 \end{pmatrix} \text{ rotated through } \tan^{-1}\frac{\Delta_4}{\Delta_5} \text{ about } x_3 \text{ is } \begin{pmatrix} K_1 \\ K_1 \\ K_1 \\ 0 \\ \sqrt{\Delta_4^2 + \Delta_5^2} \\ 0 \end{pmatrix} \text{ and rotated 45° about } x_2' \text{ is } \begin{pmatrix} K + \sqrt{\Delta_4^2 + \Delta_5^2} \\ K \\ K - \sqrt{\Delta_4^2 + \Delta_5^2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (21.24)$$

D = electric induction
E = electric field
k = dielectric constant matrix (square)
K = dielectric constant matrix (single column)
a = transformation matrix for vectors
α = transformation matrix for tensors (of stress tensor sort)
X = stress matrix (single column)
e = strain matrix (square)
ϵ = strain matrix (single column)
S = elastic modulus matrix
C = elastic constant matrix
H = temperature change of elastic modulus matrix

h = temperature change of elastic constant matrix
T_s = temperature coefficient of elastic modulus terms
T_c = temperature coefficient of elastic constant terms
A = temperature expansion matrix
d = piezo-electric constant matrix
g = inverse piezo effect matrix
G = electro-striction matrix
Z = electro-optic matrix
π = stress optic matrix
m = strain optic matrix
P = polarization = elec. mom. per unit vol. = surface charge per unit area
η = susceptibility $= \dfrac{P}{E}$

Table of Equations	Transformations
$D = \frac{1}{4\pi} kE$	The form $\begin{matrix} & x & y & z \\ x' & l_1 & m_1 & n_1 \\ y' & l_2 & m_2 & n_2 \\ z' & l_3 & m_3 & n_3 \end{matrix}$
$e = SX$	is the transformation $a = \begin{matrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{matrix}$
$X = Ce$	
$S = C^{-1}$	
$h = -C^{\circ}HC^{\circ}$	
$H = -S^{\circ}hS^{\circ}$	
$\Delta l = tAl + t^2Bl$	
$D = dX$	$A' = aAa^{-1}$
$e = gE$	$D' = aD$
$g = d_e$	$E' = aE$
$K^{-1} = K^{-1^{\circ}} + ZE$	$k' = akd_e$
$K^{-1} = K^{-1^{\circ}} + \pi X$	$K' = \alpha K$
$K^{-1} = K^{-1^{\circ}} + me$	$X' = \alpha X$
$P = \frac{k - I}{4\pi} E = \pi E$	$e' = \alpha_c^{-1} e$
$\pi = \frac{k - I}{4\pi}$	$C' = \alpha C \alpha_e$
	$S' = \alpha_c^{-1} S \alpha^{-1}$
	$h' = \alpha h \alpha_e$
	$H = \alpha_c^{-1} H \alpha^{-1}$
	$h_{ij} = C^{\circ}_{ij} T C_{ij}$
	$H_{ij} = S^{\circ}_{ij} T S_{ij}$
	$d' = ad\alpha^{-1}$

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The Metallurgy of Fillet Wiped Soldered Joints*

By E. E. SCHUMACHER, G. M. BOUTON, G. S. PHIPPS

THE seriousness of the present tin scarcity has stimulated large consumers of this vital metal to develop drastic conservation measures in order to extend the available supplies to cover the emergency period. By devising new soldering methods and alloys the Bell System has contributed a substantial share in the tin conservation effort. Fortunately, the changes, as far as can now be determined, have not introduced weakness into the soldered joints. Some of the new procedures now used were already in the process of development at the onset of the emergency, while others were devised under its stress. In some instances, the newly developed solders were found to be more difficult to use than the alloys previously available, and would not have been introduced under normal conditions. One major change made that previously had been under consideration will result in large tin savings. Unless service difficulties are encountered, this modification gives promise to remain after the emergency has passed. The change involves a reduction in the amount of solder placed on a wiped joint between the cable sheath and the sleeve. Instead of the customary full size wiped joint a wipe of fillet proportions is formed. Through this change, a solder saving of over 60% per joint can be realized.

Plumbers and cable splicers have for many years joined lead pipes and cable sheath by a soldering process called "wiping." The name is an apt description of the operation. In wiping a joint the sections to be united are heated by pouring molten solder over their surfaces and manipulating the resulting semi-liquid mass by wiping with cloth pads to a well rounded symmetrical form such as is shown in Fig. 1. The operation requires considerable skill on the part of the splicer and close control of the solder composition. At first consideration, the problem of tightness in such joints seems simple but experience shows that even under the best conditions the fissures frequently found in the solder occasionally link to form a path that allows leakage to occur. In the case of telephone cables not maintained under gas pressure, such leaks permit the entrance of water to wet the paper covered conductors, thereby impairing the insulation value and causing service interruptions. By going to an extreme and wiping off all the solder in excess of a fillet, it has been found that many causes of porosity are eliminated. Figures 2 and 3 show cross sections of joints wiped the old and new

* Reprinted from *Metals Technology*, *A.I.M.E.*, 1943.

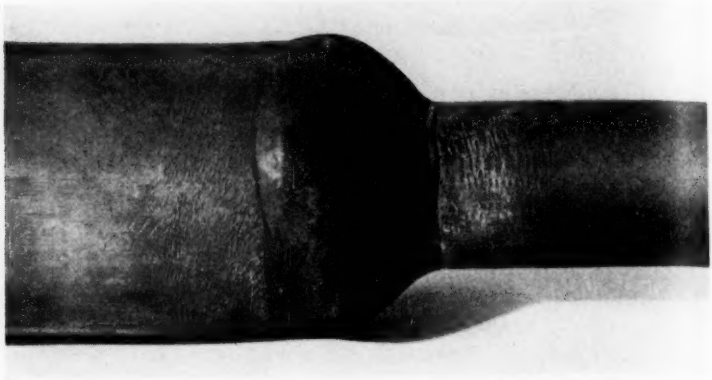

Fig. 1—A conventionally wiped joint between telephone cable and sleeve (third size).

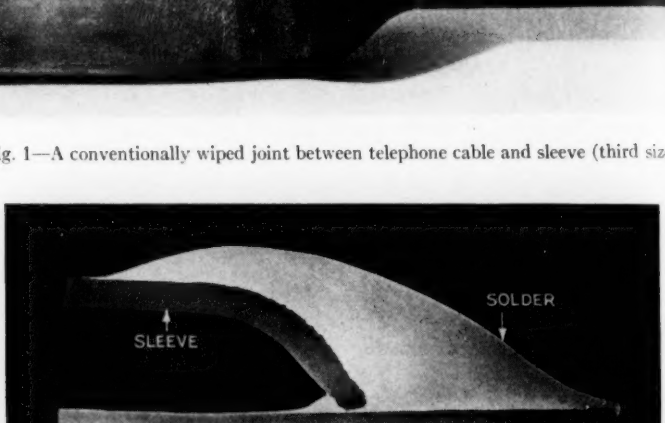

Fig. 2—A section taken from a joint wiped conventionally (magnification 1½ ×).

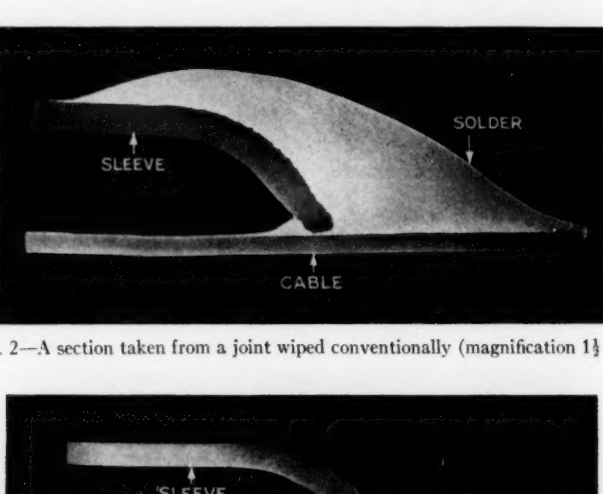

Fig. 3—A section taken from a joint wiped using the fillet technique (magnification 1½ ×).

ways. The saving in solder and consequently in strategic tin is evident. The field splicing forces find that joints are easier to make by the new method and are less apt to be porous.

Several interesting metallurgical considerations which are responsible for the success of the fillet wipe will now be discussed briefly. Much has been written about the wiping process of soldering cable joints and the many requirements of a good wiping solder have been frequently listed. The success of the procedure here described is dependent upon a few fundamental characteristics of lead-tin alloys in the process of freezing which have sound metallurgical explanations.

For an understanding of the defects possible in a soldered joint wiped in the usual manner, the simple solidification phenomena of metals may be considered. As is well known, molten metal in a crucible when allowed to cool with free circulation of air will begin freezing near the walls of the vessel and with a few exceptions, will end with a concave surface due to solidification shrinkage. Restricting the discussion to a simple lead-tin wiping solder, solidification progresses as follows: a lead-tin solid solution commences to freeze and forms a rather porous cylinder touching the crucible walls and extending to a height corresponding to the volume of the melt at that time; on further cooling, dendrites of lead-tin solid solution grow inward toward the center of the crucible and at the same time many tiny new crystals form throughout the liquid. There are thus taking place simultaneously, shrinkage of metal as it becomes solid, shrinkage of previously frozen solid as it cools, and shrinkage of the remaining liquid as the temperature drops. The originally solidified outer cylinder, adhering to the crucible walls remains essentially at its original height. The level of the semi-liquid portion nearer the center of the crucible continuously falls until the precipitated crystallites formed in the body of the melt make a loosely piled mass extending from the upper surface to the bottom of the crucible. Further shrinkage of the liquid then leaves these primary crystallites at approximately this level while the liquid recedes, leaving fissures between them. This can be beautifully observed by means of a binocular microscope focussed on the surface of a soldifying crucible of wiping solder or, on the top surface of a solidifying wiped joint.

Further insight into the mechanism of wiping solder solidification may be gained by another simple illustration. If two solder strips are cast by pouring small quantities of molten solder, one on a cold iron surface, and the other on a cloth-covered board and both are then bent cold to produce specimens as shown in Fig. 4, the chill cast sample will exhibit fewer cracks resulting from shrinkage than the slowly cooled one. In the slowly cooled sample primary crystallites form throughout the solidifying mass and pack at a level above that which the final volume of completely solid solder

warrants. The sample cast on the cold plate starts to freeze at its surface in contact with the iron plate and, because of the rapid extraction of heat by the cold iron, it continues to freeze in a rapidly advancing smooth front until the last liquid at the top is solid. Because of the steep temperature gradient there is little opportunity for nucleation and dendrite formation in the upper liquid. The surface of this melt is therefore smooth and free from the fissures that are caused by the shrinkage of the eutectic away from

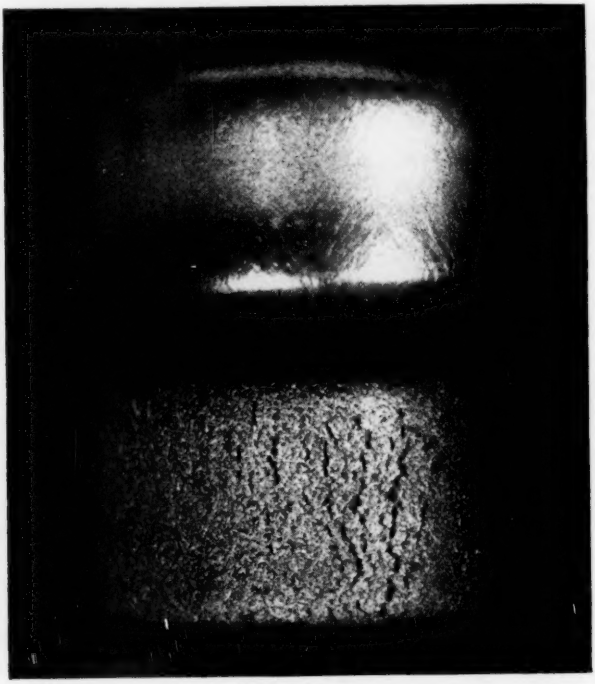

Fig. 4—Bent strips illustrating the effect of variations in cooling rate on the structure of wiping solders are here shown. The upper strip was chill cast and shows a sound ductile surface. The lower strip of the same solder was slowly cooled and upon bending exposes fissures between the crystallites at the surface (magnification 3 ×).

the dendrites in the slowly cooled sample. Recession of the liquid in the slowly cooled sample leaves a multitude of shrinkage channels which, if they occurred at the critical portion of a wiped joint, would cause leaks.

Another illustration may be useful in demonstrating the processes taking place in connection with joint wiping. Solder may be allowed to solidify in a crucible until its surface is quite firm to a probe. If, at this stage, the

crucible is tilted sideways to a position shown in Fig. 5 a portion of the remaining eutectic may be poured out leaving spongy regions. This loss of eutectic is observed frequently during the formation of the old massive type joints which may lose several drops by drainage after the splicer has completed his shaping operations. It is also shown by the greater number of pores in the top half of a joint compared to the bottom and the somewhat grayer surface appearance of the top.

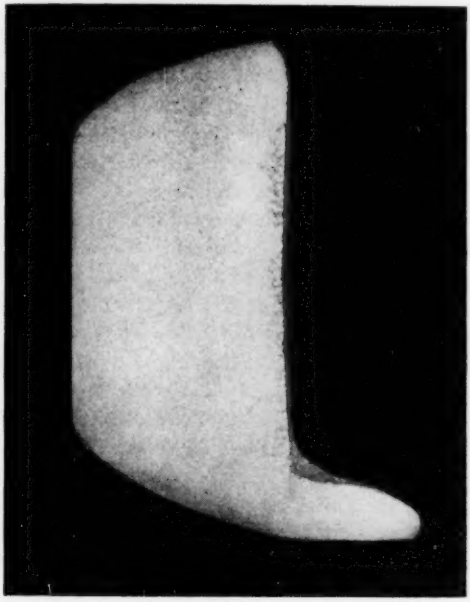

Fig. 5—An ingot of wiping solder which had been tilted while in the crucible before completely solidified. The lower lip represents eutectic drainage from the partially solidified mass (magnification 1½ ×).

Although a solidification range in which quantities of liquid and solid metal may exist at equilibrium is an essential feature of a wiping solder, another factor of major importance is the nucleation rate of the alloy. Wiping solders having high nucleation rates will develop quickly a myriad of points or nuclei throughout the melt from which further crystallization will proceed, while an alloy of low nucleation rate will develop relatively few of these points in the same time and consequently grow fewer and larger crystals. The former alloy will have a texture similar to fine clay while the latter will behave like coarse sand and water when subjected to wiping

tests. In the fine clay-like texture there are more solid particles present than in the water-sand type of texture and therefore there is more surface available for the retention of the liquid in the former type of semi-solid mass. Drainage is thus greatly retarded with the result that porosity is materially lessened. The type of texture determines in a large measure the ease of shaping and potential porosity of a wiping solder.

Having examined elementary forms of solidification, attention may now be focussed on the setting up of the wiped joint itself. In practice, the parts to be joined are cleaned and fluxed. Circumferential paper pasters are then applied to the sheath and sleeve to restrict the spread of the solder. The splicer then pours hot solder from a ladle over the prepared parts and catches the excess in a cloth held in contact with the bottom of the joint. The caught solder is repeatedly pushed back around the cable with a wiping motion to aid "tinning" or alloying and to distribute the heat. After a few such operations the prepared surfaces can be seen to be thoroughly wetted by the solder. At this stage a portion of the caught solder is mixed in the ladle with more hot solder and the mass which now has a clay-like consistency is poured on the joint and molded into place using cloth pads. When solidification has proceeded to a condition where the solder can support itself in position, manipulation is stopped. From this point on, loss of heat takes place by conduction away from the joint by the sheath and sleeve, by radiation, and by air convection currents at the surface of the solder. As a result of this combination of heat losses final solidification takes place in the interior of the solder mass near the important sheath-sleeve junction. The action that causes pipes to form in castings draws the eutectic from the critical area between the sheath and the end of the sleeve. If the solder has the proper characteristics there will be a shell of solder which does not have interconnecting shrinkage cavities, drainage cavities or fissures due to the wiping operation and the finished joints will be gas tight. If the solder is unduly coarse or has insufficient liquid eutectic at the time the mass is too rigid to manipulate further, the resulting joint may leak.

The new fillet wiping technique is similar to the old up to the step where the splicer molds the mass to shape. At this point the new technique consists in wiping the solder to a small fillet similar to that shown in Fig. 3. The resulting joint has much less solder and therefore much less total shrinkage and tendency to draw eutectic from the space between the sheath and sleeve. Also, at the temperature where wiping is discontinued there is insufficient volume of solder left by the fillet wiping technique to permit drainage drops to accumulate and fall from the bottom of the joint. Thermal conduction along the sheath and sleeve cause rapid solidification of the solder at the joint, eliminating the possibility of drainage. Experience has shown a consistently high percentage of sound joints when fillet wiping is

rigidly practiced. During the development period of the fillet wiping technique examination of the few fillet type wiped joints that were found to leak showed quantities of solder present much in excess of that required. Under the microscope such joints showed the tell-tale sponginess where the eutectic had been drawn away from the junction in the course of final solidification.

Physical tests on joints made using the fillet wipe between sections of telephone cable and sleeving, have demonstrated that fillet joints similar in size to that shown in Fig. 2 made with 38% tin, 0.1% arsenic, balance lead wiping solder are stronger in tensile strength, creep and fatigue than the cable itself.

The application of the new technique has gone much further toward saving tin than any known permissible change in the composition of solder. Using the old technique, a reduction of only one per cent in the nominal tin content of a lead-tin wiping solder resulted in widespread occurrence of leaky joints, indicating that little tin could be saved by a simple change in solder specification. This observation was to be expected since many studies had been conducted over the years to reduce the tin content in wiping solders to the minimum consistent with the production of satisfactory joints. Tin has always been much more expensive than lead and for large users of solder a reduction of one per cent in the tin content might result in savings of many thousands of dollars annually.

While the use of the fillet wipe results in large savings in tin other avenues for conserving this strategic metal are available such as the substitution of ternary and quaternary alloys containing less tin than that required by the binary lead-tin wiping solders. A satisfactory alloy of this type was developed which contains 13% tin, 23% bismuth, 0.1% arsenic, balance lead. Though readily available a short time ago, bismuth now has become too restricted to be used extensively in solders. A wiping solder is now being introduced into service in which, through the inclusion of a small quantity of antimony, it has been possible to reduce the tin content. This material appears suitable for fillet wiping although it requires more skill to use than the 38% tin, 0.1% arsenic, balance lead wiping solder. Other compositions may be usable that contain less than normal tin, but on the whole, the savings accomplished by composition modifications will be small compared to those produced by the new wiping technique that has been described.

In Summation

By virtue of its small solder volume the fillet wipe reduces tin consumption and produces joints less liable to leakage than the conventional wiped joints. The reasons for the success of this type of joint are based on the sound metallurgical principles herein described. The use of the fillet wipe promises to survive the period of restricted tin consumption.

A Mathematical Theory of Linear Arrays

By

S. A. SCHELKUNOFF

A MATHEMATICAL theory, suitable for appraising and controlling directive properties of linear antenna arrays, can be based upon a simple modification of the usual expression for the radiation intensity of a system of radiating sources. The first step in this modification is closely analogous to the passage from the representation of instantaneous values of harmonically varying quantities by real numbers to a symbolic representation of these quantities by complex numbers. The second step consists in a substitution which identifies the radiation intensity with the norm[1] of a polynomial in a complex variable. The complex variable itself represents a typical direction in space. This mathematical device permits tapping the resources of algebra and leads to a pictorial representation of the radiation intensity.

An *antenna array* is a spatial distribution of antennas in which the individual antennas are geometrically identical, similarly oriented, and energized at similarly situated points. The first and the last properties insure that the *form* of the current distribution is the same in all the elements of the array and that consequently the array is composed of antennas with the same radiation patterns. The difference between individual elements consists merely in the relative phases and intensities of their radiation fields. The second property means that the radiation patterns of the individual elements are similarly oriented and that consequently the *radiation pattern of the array is the product of the radiation patterns of its typical element and the "space factor"*. The *space factor* of an array is defined as the *radiation pattern* of a similar array of *non-directive* elements. Hence in studying the effect of spatial arrangement of antennas, we may confine ourselves to non-directive elements and thus materially simplify the analysis.

An array is *linear* if points, similarly situated on the elements, are colinear. In this paper we are concerned mostly with linear arrays of *equispaced* sources although in conclusion we shall have an occasion to say a few words about more general types.

[1] The norm of a complex number is the square of its absolute value.

Radiation Intensity and Field Strength

Consider a linear array of n equispaced nondirective sources (Fig. 1). Apart from the inverse distance factor, the instantaneous field strength of the array in the direction making an angle θ with the line of sources may be expressed as follows

$$\sqrt{\Phi_i} = A_0 \cos(\omega t + \vartheta_0) + A_1 \cos(\omega t + \psi + \vartheta_1) + A_2 \cos(\omega t + 2\psi + \vartheta_2) + \cdots + A_{n-2} \cos(\omega t + \overline{n-2}\,\psi + \vartheta_{n-2}) + \cos(\omega t + \overline{n-1}\,\psi), \quad (1)$$

$$\psi = \beta \ell \cos\theta - \vartheta, \qquad \beta = \frac{2\pi}{\lambda}.$$

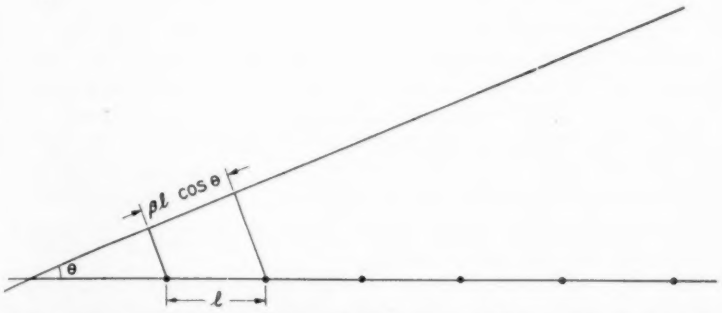

Fig. 1—A linear array of equispaced non-directive sources. If two sources are of equal intensity and in phase, their fields at a distant point are substantially equal in intensity but differ in phase by $\beta\ell \cos\theta$ where $\ell \cos\theta$ is the projection of the distance between the sources upon the particular spatial direction under consideration. If the sources are unequal, an allowance must be made for the relative field intensities in proportion to magnitudes of the sources and the phases must be adjusted for the phase difference between the sources.

In this equation: $A_0, A_1, \cdots A_{n-1} = 1$ are the relative amplitudes of the elements of the array; ϑ is a progressive phase delay, from left to right, between the successive elements of the array; $\vartheta_1, \vartheta_2, \cdots \vartheta_{n-2}, \vartheta_{n-1} = 0$ represent the phase deviations from the above progressive phase delay; $\beta = 2\pi/\lambda$ is the phase constant, where λ is the wavelength. The *radiation intensity*, that is the power radiated per unit solid angle, is proportional to the square of the amplitude of $\sqrt{\Phi_i}$.

Forming another expression similar to (1) but with sines in the place of cosines, multiplying the result by $i = \sqrt{-1}$ and adding it to (1), we have

$$\sqrt{\Phi_i'} = [A_0 e^{i\vartheta_0} + A_1 e^{i\psi + i\vartheta_1} + A_2 e^{2i\psi + i\vartheta_2} + \cdots + A_{n-2} e^{i\overline{n-2}\psi + i\vartheta_{n-2}} + e^{i\overline{n-1}\psi}]\, e^{i\omega t}. \quad (2)$$

The true instantaneous value of the field strength is the real part of (2).

Hence the amplitude $\sqrt{\Phi}$ of the field strength[2] is the absolute value of (2); thus[3]

$$\sqrt{\Phi} = \left| a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-2} z^{n-2} + z^{n-1} \right|, \qquad z = e^{i\psi}, \psi = \beta \ell \cos\theta - \vartheta, a_m = A_m e^{i\vartheta m}. \tag{3}$$

In this equation: a_0 , a_1 , a_2 , $\cdots$ a_{n-2} , $a_{n-1} = 1$ are complex numbers representing the relative amplitudes of the elements of the array and the phase deviations of these elements from a given progressive phasing. Thus if all the coefficients are real and positive, they represent the relative amplitudes of the elements of the array. If the algebraic sign of a particular coefficient is reversed, the phase of the corresponding element is changed by 180°; if some coefficient is multiplied by i or $-i$, the phase of the corresponding element is respectively accelerated or delayed by 90°; and in general the phase acceleration is equivalent, in our scheme, to a multiplication by a unit complex number $e^{i\vartheta}$. Some coefficients may be equal to zero and the corresponding elements of the array will be missing. In view of this possibility, we shall call ℓ the "apparent" separation between the elements; it is the greatest common measure of actual separations. When the elements are equispaced the apparent separation is the actual separation.

Thus we have the fundamental

Theorem I: *Every linear array with commensurable separations between the elements can be represented by a polynomial and every polynomial can be interpreted as a linear array.*[4]

The total length of the array is the product of the apparent separation between the elements and the degree of the polynomial. The degree of the polynomial is one less than the "apparent" number of elements. The actual number of elements is at most equal to the apparent number.

The above analytical representation of arrays is accomplished with the aid of the following transformation

$$z = e^{i\psi}, \tag{4}$$

in which $\psi = \beta \ell \cos\theta - \vartheta$ is a function of the angle θ made by the line of sources with a typical direction in space. Since ψ is always real, the absolute value of z equals unity and z itself *is always on the circumference of the unit circle* (Fig. 2). As θ increases from 0° (which is in a direction of the line of sources) to 180° (which is in the opposite direction), ψ decreases and

[2] For brevity's sake, we shall call $\sqrt{\Phi}$ itself the "field strength."

[3] Equation (3) could be derived directly from the physics of the situation in the same manner as (1). The foregoing method of transition from (1) to (3) serves only the purpose of showing the relationship between a less familiar formula and a very well known one.

[4] If the separations are not commensurable the arrays are represented by an *algebraic* function with incommensurable exponents.

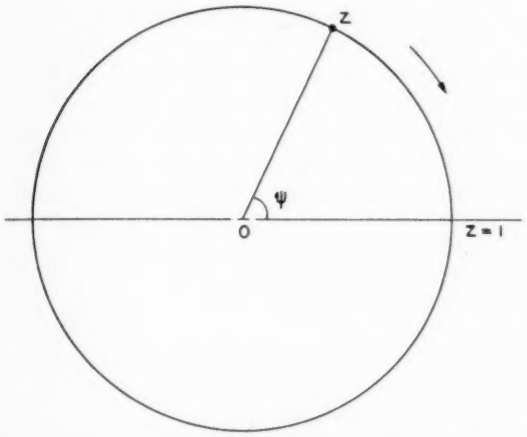

Fig. 2—A typical direction in space is represented by a complex variable which is represented in a complex plane by a point lying on the circumference of a circle of unit radius, having its center at the origin. As the angle θ made by a typical direction with the line of sources, increases from 0° to 180°, point z moves clockwise.

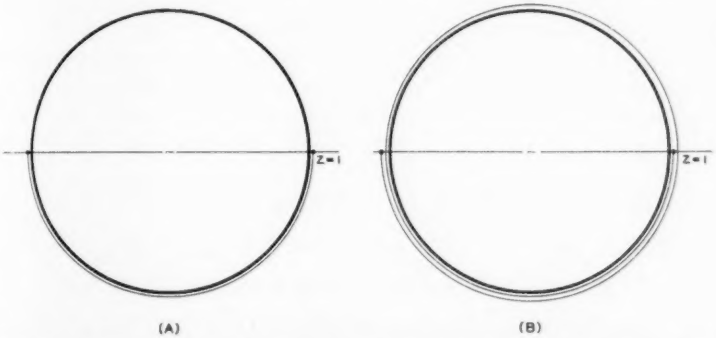

Fig. 3—(A) The active range of z, corresponding to $\vartheta = \beta\ell$ and one-quarter wave-length separation between the elements. (B) The active range of z, corresponding to $\vartheta = \beta\ell$ and $\ell = \frac{3}{4}\lambda$.

z moves in the clockwise direction. When $\theta = 0$, $\psi = \beta\ell - \vartheta$; and when $\theta = 180°$, $\psi = -\beta\ell - \vartheta$. Hence the range $\bar{\psi}$ described by z is

$$\bar{\psi} = 2\beta\ell. \tag{5}$$

When the separation ℓ between the successive elements of the array is equal to one-half wavelength, the range of $z = 2\pi$ and as θ varies from 0° to 180°, z describes a complete cycle and returns to its original position.

In this case there is a one-to-one correspondence between the points of the circumference of the unit circle and conical surfaces coaxial with the line of sources. Such conical surfaces, called radiation cones, are loci of directions in which the radiation intensities are equal. If the separation between the elements $< \lambda/2$, the range of z is smaller than 2π and z describes only a portion of the unit circle (Fig. 3A). Finally, if $\ell > \lambda/2$, then the path of z overlaps itself (Fig. 3B). Such a path, winding upon itself, will be called a *Riemann circle*. In this instance, one and the same point on the circle may correspond to several radiation cones; but if we regard different positions of z along its path as distinct points on the Riemann circle, then there will be a one-to-one correspondence between the points on the circle and the radiation cones.

Since the radiation intensity is a periodic function of ψ, the space factor of a given array will repeat itself if the separation between the elements is greater than one-half wavelength.

Composition of Space Factors

Since the product of two polynomials is a polynomial, we obtain the following corollary to Theorem I

Theorem II: *There exists a linear array with a space factor equal to the product of the space factors of any two linear arrays.*

In other words, there is a linear array such that its radiation intensity in any given direction is the product of the radiation intensities in this direction of any two given arrays. Thus we have

$$\begin{aligned} \sqrt{\Phi_1} &= |\, a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} \,|, \\ \sqrt{\Phi_2} &= |\, b_0 + b_1 z + b_2 z^2 + \cdots + b_{m-1} z^{m-1} \,|, \\ \sqrt{\Phi_1}\,\sqrt{\Phi_2} &= |\, (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1})(b_0 + b_1 z + \cdots + b_{m-1} z^{m-1}) \,| \\ &= |\, a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots \,|. \end{aligned} \tag{6}$$

The coefficients of the expanded product represent the amplitudes and the phases of the derived array.

Naturally the process may be repeated and a linear array can be constructed with its space factor equal to the product of the space factors of any number of linear arrays or to any power of the space factor of any array.

For example, let us start with a pair of equal sources, represented by

$$\sqrt{\Phi} = |\, 1 + z \,|, \tag{7}$$

and construct a linear array with the space factor equal to the square of (7). The field strength of the required array will be

$$\sqrt{\Phi} = |\, 1 + z \,|^2 = |\, 1 + 2z + z^2 \,|. \tag{8}$$

This array consists of three elements with amplitudes proportional to 1, 2, 1. If the elements of the original couplet are one-quarter wavelength apart and 90° out of phase, the couplet is "unidirectional". The space factor of such a couplet is depicted by Curve A in Fig. 4. The space factor of the triplet represented by (8) is shown by Curve B. In the directions in which the couplet radiates half as much or a third as much power as in the principal direction, the triplet radiates correspondingly only a quarter or a ninth of the power radiated in the principal direction.

The above considerations suggest a simple method for suppressing subsidiary radiation lobes. It is well known that in a uniform linear array[5]

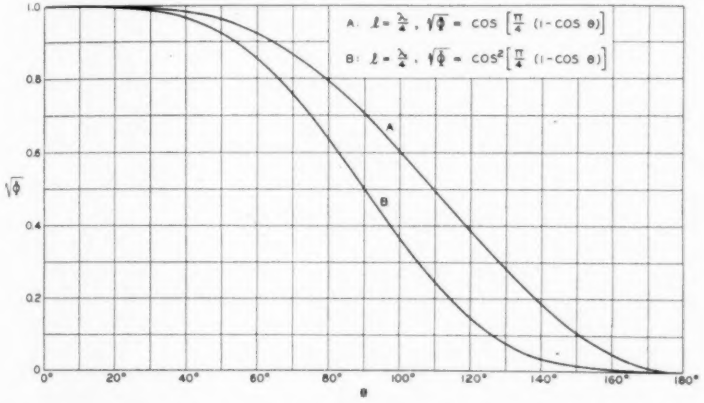

Fig. 4—Space Factors—Curve A is the space factor of a unidirectional couplet in which $\ell = \lambda/4$. Curve (B) represents the space factor of an array with amplitudes proportional to 1, 2, 1.

the difference in levels of the principal maximum of radiation and the first subsidiary is substantially independent of the number of elements, provided this number is sufficiently large. Thus in the limit, the first subordinate maximum is 13.5 decibels below the principal maximum. Consequently for the array with its space factor equal to the square of the space factor of the uniform array, the limiting difference in levels must be 26.9 decibels.

Since the uniform array is represented by

$$\sqrt{\Phi} = |\, 1 + z + z^2 + \cdots + z^{n-1} \,|, \tag{9}$$

[5] A "uniform" array is an array made up of sources of equal strength with a uniform progressive phase delay.

the other array is given by

$$\sqrt{\Phi} = |1 + z + \cdots + z^{n-1}|^2$$
$$= |1 + 2z + 3z^2 + \cdots + nz^{n-1} + (n-1)z^n + \cdots + 2z^{2n-3} + z^{2n-2}|. \quad (10)$$

Thus the amplitudes of the individual sources are proportional to 1, 2, 3, $\cdots$ $n-1$, n, $n-1$, $\cdots$ 3, 2, 1. Figure 5 depicts the effect of such "triangular" amplitude distribution.

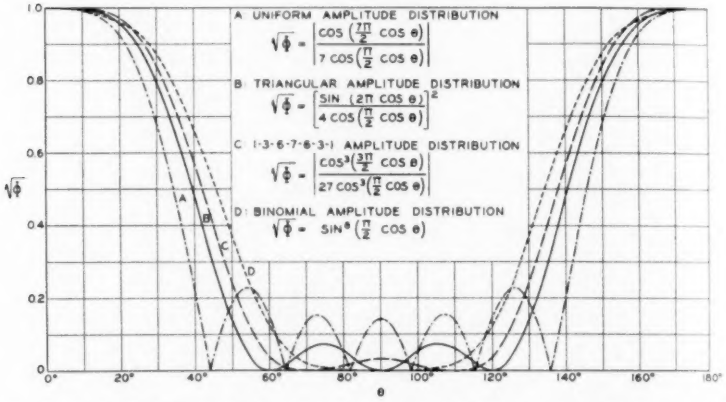

Fig. 5—Space Factors—(A) is for a uniform array and (B) for an array with "triangular" amplitude distribution.

Evidently we could raise (9) to any given power

$$\sqrt{\Phi} = |1 + z + z^2 + \cdots + z^{n-1}|^m. \quad (11)$$

This process does not change the number of separate radiation lobes. The so-called "binomial" distribution of amplitudes was first suggested by John Stone Stone.[6] His scheme is a special case of (11) if we let $n = 2$. For the effect of the binomial amplitude distribution see Fig. 5.

The relative merits of two forms for the radiation intensity as given by (1) and (3) can now be appraised in the light of the foregoing examples. Using (1), we have for the instantaneous radiation intensity of the unidirectional couplet

[6] U. S. Patents 1,643,323 and 1,715,433.

$$\sqrt{\Phi_i} = \cos \omega t + \cos\left(\omega t + \frac{\pi}{2}\cos\theta - \frac{\pi}{2}\right)$$
$$= \cos \omega t + \sin\left(\omega t + \frac{\pi}{2}\cos\theta\right). \tag{12}$$

By just inspecting this equation, we find no evidence for existence of a linear array with a space factor equal to the square of the space factor of the couplet. Still less obvious is the method of obtaining proper amplitude ratios.

Arrays of Arrays

The foregoing method of composition of space factors is in reality an analytical expression of geometric construction of "arrays of arrays". Consider, for instance, a pair of equiphase sources of equal strengths

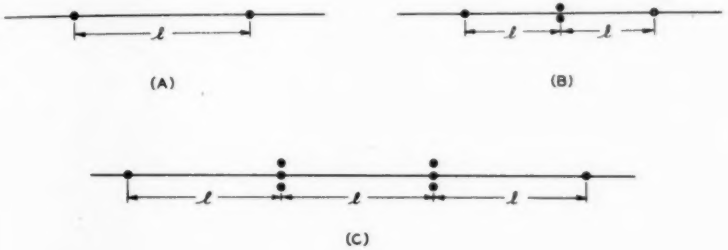

Fig. 6

(Fig. 6A). Take two such pairs as elements of an array of the same type (Fig. 6B). The middle sources add up to a single source of strength two. If the operation is repeated by taking (B) as elements of (A) or by taking (A) as elements of (B), then (C) is obtained; the amplitudes of (C) are proportional to 1, 3, 3, 1.

Each shift of a source to the right through distance ℓ is represented analytically as multiplication by z. An algebraic identity

$$(a_0 + a_1 z + a_2 z^2)z = a_0 z + a_1 z^2 + a_2 z^3 \tag{13}$$

is an expression of an obvious fact that each element of an array is shifted through the same distance as the entire array. Similarly a given change in the strength and the phase of the array is achieved by making the same change in all its elements; this fact is expressed by the identity

$$b(a_0 + a_1 z + a_2 z^2) = ba_0 + ba_1 z + ba_2 z^2. \tag{14}$$

In general, if an array represented by

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} \tag{15}$$

is taken as the element of an array given by

$$F(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_{m-1} z^{m-1}, \tag{16}$$

then the resulting array of arrays is represented by

$$f(z)F(z) = b_0 f(z) + b_1 z f(z) + b_2 z^2 f(z) + \cdots + b_{m-1} z^{m-1} f(z). \tag{17}$$

Decomposition Theorem

Consider now a pair of non-directive sources with strengths proportional to 1, $-t$; then

$$\sqrt{\Phi} = |\, z - t \,|. \tag{18}$$

Geometrically, the complex number $z - t$ is represented by a line drawn from point t to point z (Fig. 7A). Accordingly, the *radiation intensity*

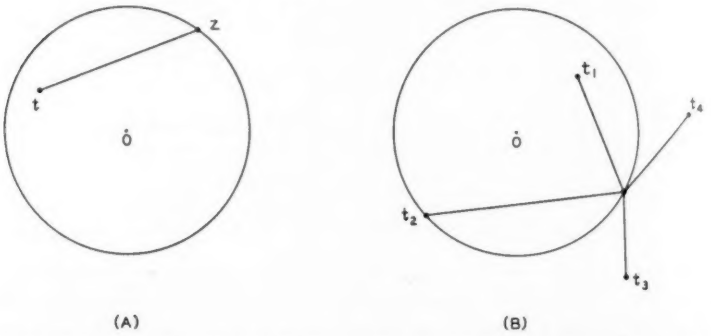

Fig. 7—The radiation intensity of a linear array is represented by the square of the product of the lines joining the null points of $\sqrt{\Phi}$ to a point z on the unit circle.

of the pair of sources *is represented by the distance between t and z.* If $\sqrt{\Phi}$ vanishes for some particular direction in space, it vanishes for all directions making the same angle with the line of sources; these directions form a *cone of silence* of the radiation system. Obviously, a radiating couplet has a cone of silence if and only if the null point of $\sqrt{\Phi}$ is in the range of z; in particular, there can be no cone of silence unless the null is on the unit circle.

By the fundamental theorem of algebra a polynomial of degree $(n - 1)$ has $(n - 1)$ zeros (some of which may be multiple zeros) and can be factored into $(n - 1)$ binomials; thus

$$\sqrt{\Phi} = |\, (z - t_1)(z - t_2) \cdots (z - t_{n-1}) \,|. \tag{19}$$

Each binomial represents the directive pattern of a pair of elements separated by distance ℓ. Hence

Theorem III: *The space factor of a linear array of n apparent elements is the product of the space factors of* $(n-1)$ *virtual couplets with their null points at the zeros of* $\sqrt{\Phi}$: $t_1, t_2, \cdots t_{n-1}$.

Accordingly the radiation intensity of an array is equal to the square of the product of the distances from the null points of the array to that point z on the unit circle which corresponds to the chosen direction (Fig. 7B). To each null point lying in the range of z, there corresponds one and only one *cone of silence* provided each null point is counted as many times as z happens to pass it in describing the complete range.

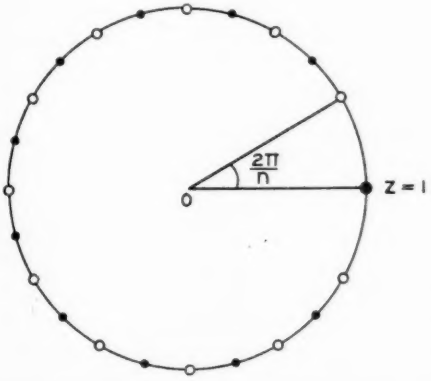

Fig. 8—The null points of a uniform linear array and the point $z = 1$ representing the direction of the greatest radiation divide the unit circle into equal parts. The hollow circles represent the null points and the solid circles the points of maximum radiation.

By summing the geometric progression (9) the radiation intensity of a uniform array can be represented as follows

$$\sqrt{\Phi} = \left| \frac{z^n - 1}{z - 1} \right|. \tag{20}$$

Hence the null points of such an array are the n-th roots of unity, excluding $z = 1$. Since z is a unit complex number,[7] any power of it is also a unit complex number. Moreover, each multiplication by $z = e^{i\psi}$ represents a displacement through an arc of ψ radians. Hence the n-th roots

[7] A *unit complex number* is a complex number whose absolute value is equal to unity.

of unity divide the circle into n equal parts (Fig. 8). Analytically we have

$$z^n - 1 = 0, \qquad t_m = e^{-\frac{2m\pi i}{n}}, \qquad m = 1, 2, 3, \cdots n - 1,$$
$$\psi_m = -\frac{2m\pi}{n}, \qquad \cos\theta_m = \frac{\vartheta}{\beta\ell} - \frac{2m\pi}{n\beta\ell}. \tag{21}$$

When $z = 1$, $\sqrt{\Phi}$ is evidently a *principal* maximum. Other maxima of smaller magnitude, the so-called *subordinate* or *subsidiary* maxima, occur approximately half way between the null points. The general

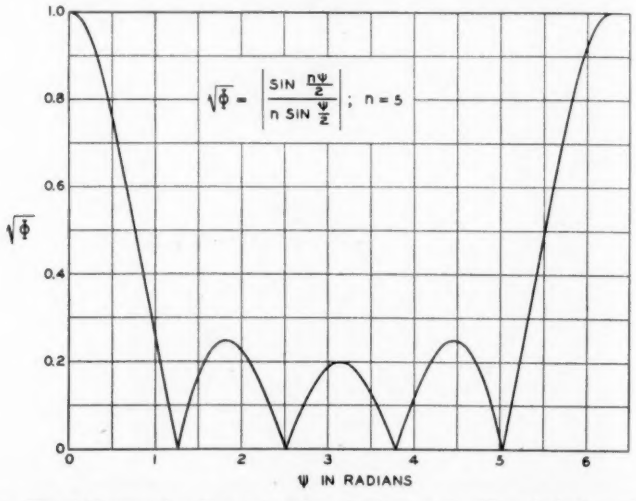

Fig. 9—The field strength $\sqrt{\Phi}$ as a function of ψ for $n = 5$. The principal maximum is reduced to unity.

behavior of the field strength can readily be understood if we follow z around the unit circle. When plotted against ψ, $\sqrt{\Phi}$ has the shape shown in Fig. 9. This is a universal radiation characteristic which can be interpreted for any particular spacing and phasing between the elements with the aid of the curve for $\psi + \vartheta = \beta\ell \cos\theta$ (Fig. 10).

It is easy to estimate the relative level of the first subordinate maximum. For a fairly large number of elements, the difference in levels is determined largely by the distances of the maximum points from the *nearest* null points. The distances are approximately equal to the circular arcs joining the corresponding points. Since the arcs joining the first subordinate maximum with the nearest null points are nearly half as long as

those for the principal maximum, the first subordinate maximum of the field strength is about one-quarter of the principal maximum. In other words, the subordinate maximum is approximately 12 decibels below the principal maximum.

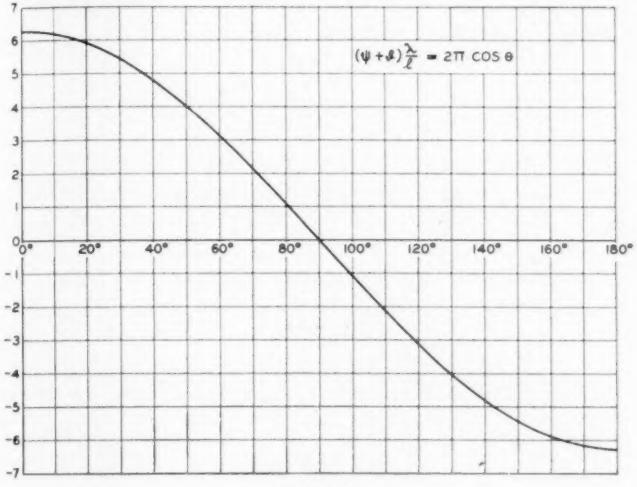

Fig. 10

A more accurate value for this difference in levels can be obtained by first rewriting (20) in the form

$$\sqrt{\Phi} = \left| \frac{z^{\frac{n}{2}} - z^{-\frac{n}{2}}}{z^{\frac{1}{2}} - z^{-\frac{1}{2}}} \right| = \frac{\sin \frac{n\psi}{2}}{\sin \frac{\psi}{2}} \tag{22}$$

and then substituting successively $\psi = 0$ and $\psi = \frac{3\pi}{n}$, one for the principal maximum and the other for the first subordinate. Accordingly we obtain

$$\frac{\sqrt{\Phi(0)}}{\sqrt{\Phi\left(\frac{3\pi}{n}\right)}} = n \sin \frac{3\pi}{2n}. \tag{23}$$

If n is large $\sin \frac{3\pi}{2n}$ is approximately equal to $\frac{3\pi}{2n}$ and the field strength ratio becomes $\frac{3\pi}{2} = 4.71$. This ratio corresponds to the difference in levels equal to 13.5 decibels.

Directivity of Arrays

The "decomposition theorem" of the preceding section throws considerable light on directive properties of arrays. The number of elements in the array is one greater than the number of virtual couplets. Hence to secure the greatest possible directivity with a given number of elements, the virtual couplets must be properly combined.

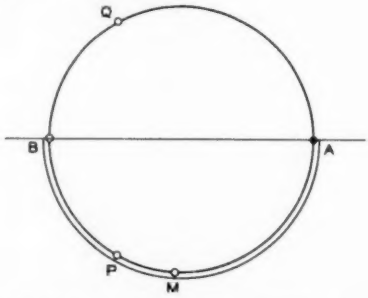

Fig. 11—The null points of several three-element arrays. The spacing between the elements is $\lambda/4$ and the progressive phase delay is $T/4$ (T equals period).

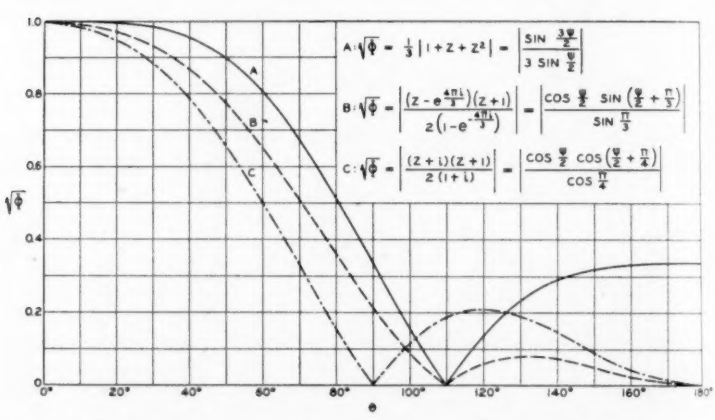

Fig. 12—Comparison of directivity of several three-element arrays. The spacing between the elements is $\lambda/4$; the direction of principal radiation is $\theta = 0°$. Curve (A) refers to the uniform array, (B) to an array with nulls at P and B (see Fig. 11), and Curve (C) refers to an array with its nulls at B and M.

For example, the null points of a uniform array of three elements, one-quarter wavelength apart, are at P and Q (Fig. 11). If $\vartheta = \pi/2$, the range of z consists of the lower half of the unit circle and principal radiation takes place in the direction $\theta = 0$. Evidently, the virtual couplet with its null

at Q is comparatively nondirective. Substituting for this couplet another couplet with a null at B should improve the directivity of the array. This is indeed the case: In Fig. 12, Curve A depicts directive properties of the uniform array and Curve B depicts those of an array with its nulls at P and B. The field strength of the second array is

$$\begin{aligned}\sqrt{\Phi} &= \left|(z - e^{-\frac{2\pi i}{3}})(z+1)\right| = \left|z^2 + (1 - e^{-\frac{2\pi i}{3}})z - e^{-\frac{2\pi i}{3}}\right| \\ &= \left|1 + z\sqrt{3}\, e^{-\frac{i\pi}{6}} + z^2 e^{-\frac{i\pi}{3}}\right|, \quad z = \frac{\pi}{2}(\cos\theta - 1);\end{aligned} \tag{24}$$

hence the amplitudes of the elements are proportional to 1, $\sqrt{3}$, 1 and the total progressive phase delay in the direction of maximum radiation is $\frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3}$ radians.

The minor lobe of the second array is substantially smaller than that of the first array. The major lobes, however, are equally "wide"[8] although one lobe is somewhat sharper than the other. The width of the major lobe can be reduced at the expense of increasing the minor lobe by moving the null from P to M (Fig. 11). The effect of this change is shown by Curve C (Fig. 12). The corresponding field strength is[9]

$$\begin{aligned}\sqrt{\Phi} &= |(z+i)(z+1)| = |z^2 + (1+i)z + i| \\ &= |1 - i(1+i)z - iz^2| = \left|1 + \sqrt{2}\, e^{-\frac{i\pi}{4}} z + e^{-\frac{i\pi}{2}} z^2\right|;\end{aligned} \tag{25}$$

hence the amplitudes are proportional to 1, $\sqrt{2}$, 1 and the total progressive phase delay is $\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$.

For arrays of six elements, one-quarter wavelength apart and with $\vartheta = \pi/2$, we have Fig. 13. Curve A represents the directive characteristic of a uniform array, with its nulls as shown in Fig. 14A, and Curve B shows the directive properties of an array with its nulls equispaced on the lower half of the unit circle as shown in Fig. 14B.

If the spacing between the elements is $\ell = \lambda/8$ and if the phase delay $\vartheta = \pi/2$, then the effect of distribution of the null points is even more pronounced (Figs. 15 and 16). This time z is confined to the fourth quadrant of the unit circle. In Fig. 15, $n = 3$; Curve A corresponds to an array with equal amplitudes in which case the nulls are equispaced on the complete unit circle (Fig. 17A) and Curve B corresponds to an array with its nulls equi-

[8] If the "width" of a lobe is measured by the angle of the cone of silence enclosing the lobe.

[9] When transforming the expressions for $\sqrt{\Phi}$, it is well to remember that the absolute value of a complex quantity does not change if this quantity is multiplied by a unit complex number.

spaced within the range of z (Fig. 17B). In Fig. 16, $n = 6$; Curve A represents an array with nulls distributed evenly on the complete circle and Curve B represents an array with nulls evenly spaced in the range of z.

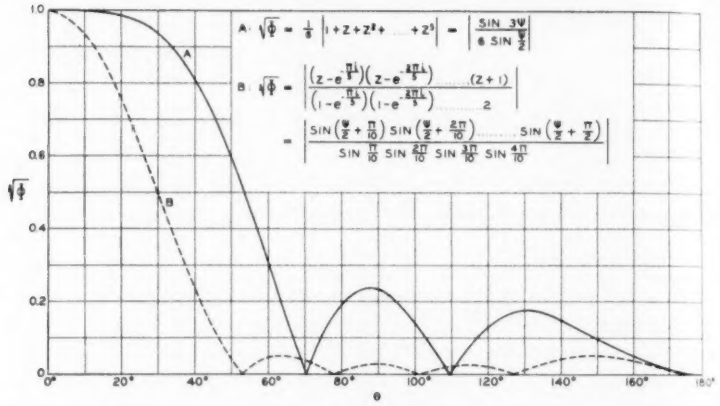

Fig. 13—Directive properties of 2 six-element arrays with $\ell = \lambda/4$. Curve (A) refers to a uniform array and Curve (B) refers to an array with its nulls equispaced in the range of z.

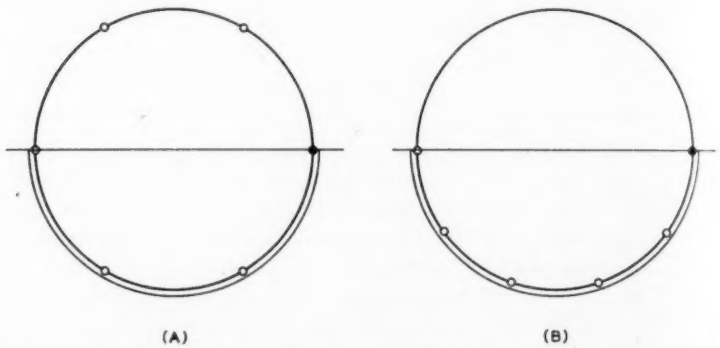

Fig. 14—Disposition of null points for the arrays with directive characteristics as shown in Fig. 13.

If the total length of an array is kept constant but the number of elements is increased, the array may be made more directive; Figure 18 illustrates this point. This increase in directivity can be secured only if the null points of the array are properly distributed within the range of z; in Fig. 18

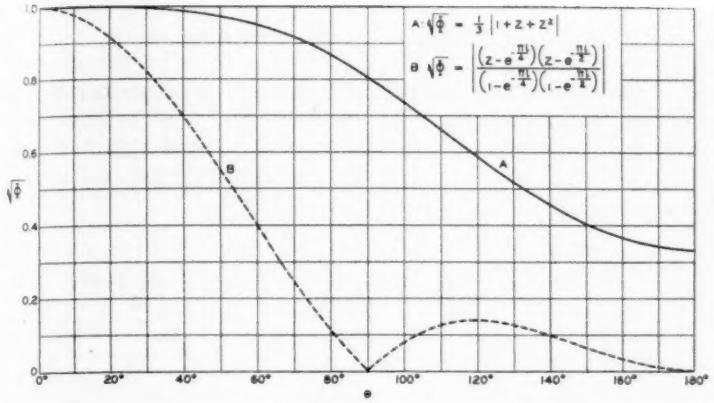

Fig. 15—Directive properties of three-element linear arrays with $\ell = \lambda/8$. Curve (A) refers to a uniform array and Curve (B) to an array with its nulls equispaced in the range of z.

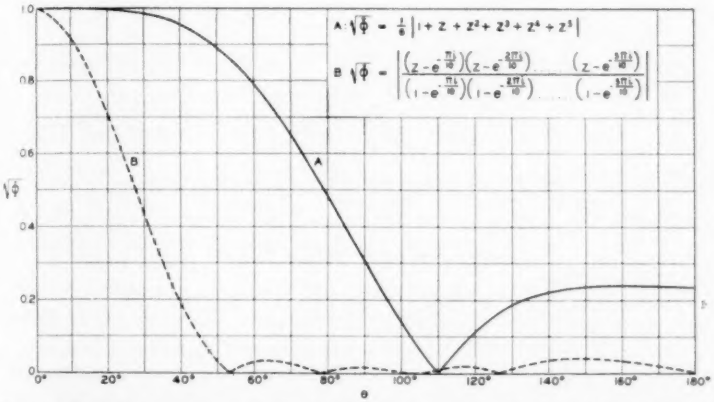

Fig. 16—Directive properties of six-element linear arrays with $\ell = \lambda/8$. Curve (A) refers to a uniform array and Curve (B) to an array with its nulls equispaced in the range of z.

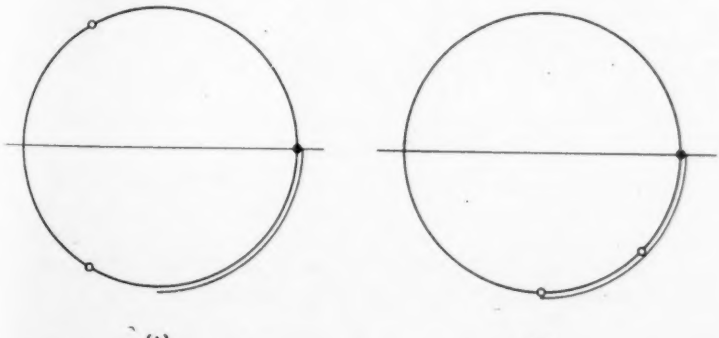

Fig. 17—Disposition of nulls for the arrays whose directive properties are shown in Fig. 15.

the null points are evenly spaced in the range of z, appropriate to each separation between the elements.

If the elements of the array are directive, the null points should be distributed with due reference to the directive pattern of the elements in order that a further increase in directivity could be secured.

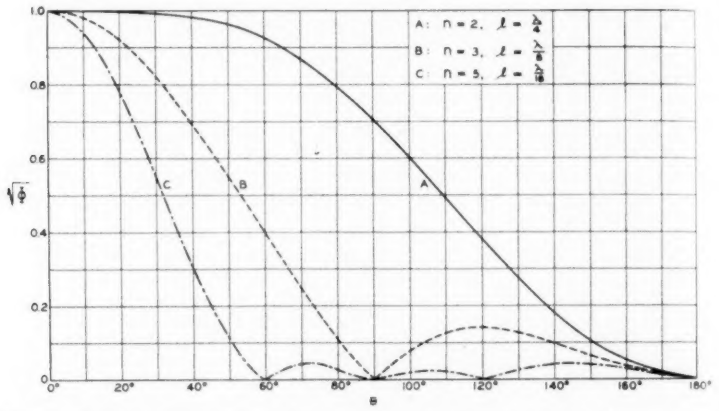

Fig. 18—Directive properties of linear arrays with total length equal to $\lambda/4$. (A), $n = 2$, $\ell = \lambda/4$; (B), $n = 3$, $\ell = \lambda/8$; (C), $n = 5$, $\ell = \lambda/16$.

Multi-Dimensional Arrays

The simplest method of constructing multi-dimensional arrays is to take a linear array as an element of another linear array. The axis of the second array may be chosen to make any angle with the axis of the first array. In this way only a special class of multi-dimensional arrays can be formed. Analytical expressions for the radiation intensities of more general arrays can be formulated in terms of two or more complex variables. These variables, however, will not be independent and a given direction in space will be represented by a group of related points, one point on each circle representing the particular complex variable. At this time we shall not be concerned with any developments applicable to such general multi-dimensional arrays.

Arrays with Prescribed Space Factors

If the minimum separation between the elements does not exceed $\lambda/2$, it is theoretically possible to design a linear array with a space factor given by an arbitrary function $f(\psi)$ or $F(\theta)$ of direction of radiation. Naturally the number of required elements will be usually infinite; with a finite number of elements the space factor may only be approximate.

Consider an array with an *odd* number of elements $n = 2m + 1$. Since the modulus of z is unity the polynomial (3) can be divided by z^m without affecting $\sqrt{\Phi}$; thus

$$\sqrt{\Phi} = |\, a_0 z^{-m} + a_1 z^{-m+1} + a_2 z^{-m+2} + \cdots + a_m + a_{m+1} z + \cdots + a_{2m} z^m \,|. \tag{26}$$

Let us now assume that the coefficients equidistant from the ends of the polynomial are conjugate complex; then the polynomial is real and we can drop the bars. Thus setting

$$a_m = A_0, \qquad a_{m+k} = A_k - iB_k, \qquad k > 0, \qquad a_{m-k} = a^*_{m+k}, \tag{27}$$

where the A's and B's are real; we have

$$\begin{aligned} a_{m+k} z^k + a_{m-k} z^{-k} &= (A_k - iB_k) e^{ik\psi} + (A_k + iB_k) e^{-ik\psi} \\ &= 2A_k \cos k\psi + 2B_k \sin k\psi. \end{aligned} \tag{28}$$

Consequently, (26) becomes

$$\sqrt{\Phi} = \sum_{k=0}^{m} \epsilon_k (A_k \cos k\psi + B_k \sin k\psi), \tag{29}$$

where ϵ_k is the Neumann number.[10]

If now we wish $\sqrt{\Phi}$ to be a prescribed function $f(\psi)$ of the variable ψ, we need only expand this function in a Fourier series

$$\sqrt{\Phi} = f(\psi) = \sum_{k=0}^{\infty} \epsilon_k (p_k \cos k\psi + q_k \sin k\psi), \tag{30}$$

and approximate it with any desired accuracy by means of a finite series (29). Once the A's and B's are known, we calculate the a's from (27).

It must be remembered that the real independent variable is not ψ but θ and the directive pattern is to be assigned as a function of θ. Besides being dependent on θ, ψ is a function of the distance ℓ between the successive elements of the array. Since θ varies from 0° to 180°, the range of ψ is $\bar{\psi} = 2\beta\ell$. The function $f(\psi)$ is prescribed within this range. On the other hand the period of the expressions (29) and (30) is 2π. This means that if $\bar{\psi} > 2\pi$, that is if $\ell > \lambda/2$, it is impossible to obtain the desired directive pattern with our scheme, because the pattern repeats itself automatically as ψ increases or decreases by 2π. But if $\ell < \lambda/2$, we have a considerable latitude in the design; outside the range of ψ, we can supplement $f(\psi)$ by an arbitrary function of ψ. It is only when $\ell = \lambda/2$ that there is a unique class of linear arrays that will produce a directive pattern given by the first

[10] $\epsilon_0 = 1$, $\epsilon_k = 2$ when $k \neq 0$.

$(m + 1)$ terms of (30). Dr. T. C. Fry of these Laboratories has suggested that leaving ℓ undetermined and fixing the number of elements, an array could be designed which would have the best fit to the prescribed pattern. In this connection, the "best fit" means the least mean square deviation of the approximating pattern from the given pattern.

If $\sqrt{\Phi}$ is given as a function $F(\theta)$ of θ, then by virtue of the definition of ψ we can write

$$F(\theta) = F\left(\cos^{-1}\frac{\psi + \vartheta}{\beta\ell}\right) = f(\psi). \tag{31}$$

Let us now consider a simple example for the sake of illustrating the method. Let $f(\psi)$ be defined by

$$\begin{aligned} f(\psi) &= 0, \qquad 0 < \psi < \pi, \\ &= 1, \qquad \pi < \psi < 2\pi. \end{aligned} \tag{32}$$

We shall assume that the separation between the elements is one-half wavelength. This makes the range of ψ equal to 2π. It is also seen that regarded as a function of θ, $f(\psi)$ retains its essential characteristic: being equal to zero over one-half of the range and to unity over the remaining half.

Expanding (32) into a Fourier series we have

$$f(\psi) = \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)\psi}{2k-1}. \tag{33}$$

Consequently

$$\begin{gathered} A_0 = \tfrac{1}{2},\ A_k = 0 \text{ if } k \neq 0; \\ B_k = 0, \text{ if } k \text{ is even}; \\ B_k = -\frac{1}{k\pi}, \text{ if } k \text{ is odd}. \end{gathered} \tag{34}$$

Figure 19 shows several approximations to $f(\psi)$ by means of a finite number of elements. The curve S_m corresponds to an approximation by the finite series (29). If S_9 is deemed to be a sufficiently good approximation to the given directive pattern, then

$$\begin{aligned} \sqrt{\Phi} = \frac{1}{\pi}\Big|\frac{1}{9} + \frac{1}{7}z^2 + \frac{1}{5}z^4 + \frac{1}{3}z^6 + z^8 + \frac{i\pi}{2}z^9 \\ - z^{10} - \frac{1}{3}z^{12} - \frac{1}{5}z^{14} - \frac{1}{7}z^{16} - \frac{1}{9}z^{18}\Big|. \end{aligned} \tag{35}$$

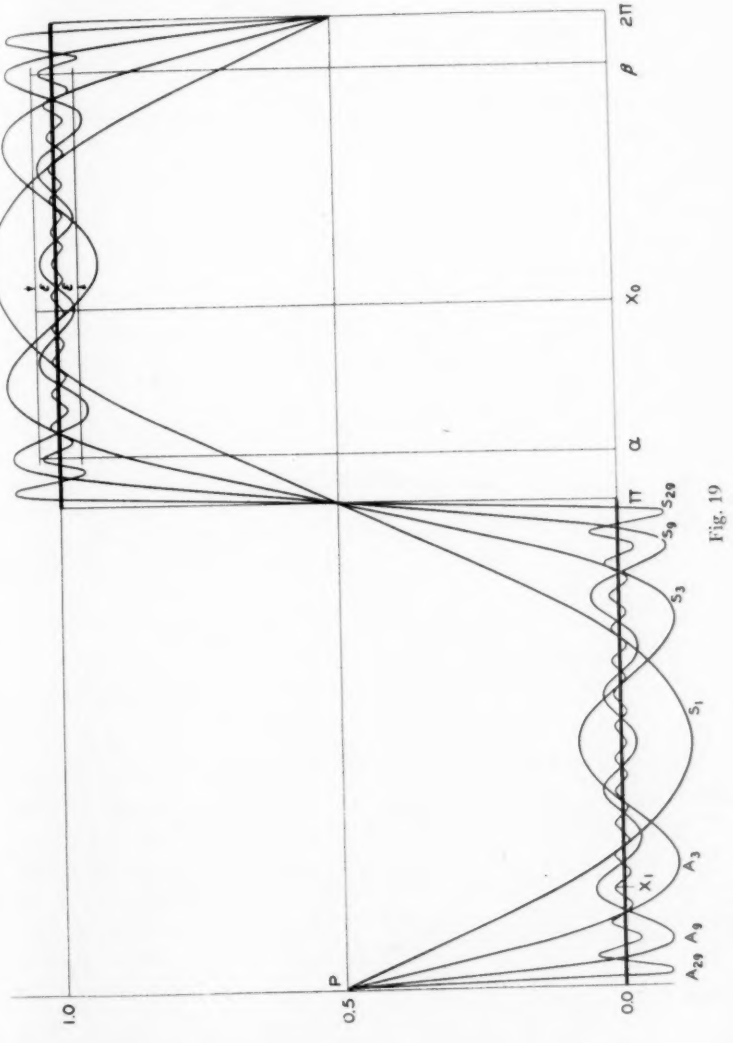

Fig. 19

The total length of this array is $(n-1)\frac{\lambda}{2} = 2m\frac{\lambda}{2} = 9\lambda$. All elements except the three central ones are separated by one wavelength since the odd powers of z except z^9 are missing.

End-On Arrays with Equispaced Null Points

We now pass to a more detailed analysis of end-on arrays with null points equispaced on a given circular arc.

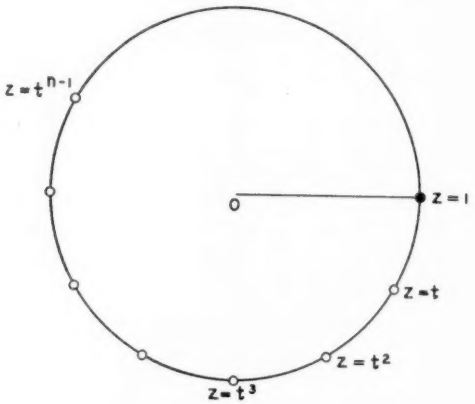

Fig. 20

For an end-on array $\vartheta = \beta\ell$ and

$$z = e^{i\psi}, \qquad \psi = \beta\ell(\cos\theta - 1). \tag{36}$$

The range of z begins at $z = 1$ and extends clockwise to a point determined by $\psi = -2\beta\ell$. Let $n-1$ null points be equispaced on an arc of length $\bar{\psi}$ as shown in Fig. 20; the field strength is then

$$\sqrt{\Phi} = \left| (z-t)(z-t^2)\cdots(z-t^{n-1}) \right|, \qquad t = e^{-\frac{i\bar{\psi}}{n-1}}. \tag{37}$$

This can be expressed as

$$\sqrt{\Phi} = 2^{n-1}\left| \sin\frac{1}{2}\left(\psi + \frac{\bar{\psi}}{n-1}\right) \times \sin\frac{1}{2}\left(\psi + \frac{2\bar{\psi}}{n-1}\right)\cdots \sin\frac{1}{2}\left(\psi + \frac{n-1}{n-1}\bar{\psi}\right)\right|. \tag{38}$$

The angle of the cone of silence enclosing the major radiation lobe is determined from

$$\beta\ell(\cos\theta_1 - 1) = -\frac{\bar{\psi}}{n-1}; \tag{39}$$

thus

$$1 - \cos\theta_1 = \frac{\bar{\psi}}{(n-1)\beta\ell}, \qquad \sin\frac{\theta_1}{2} = \sqrt{\frac{\bar{\psi}}{2(n-1)\beta\ell}}. \tag{40}$$

If the arc $\bar{\psi}$ is equal to the range of z, then (40) becomes

$$1 - \cos\theta_1 = \frac{2}{n-1}, \qquad \sin\frac{\theta_1}{2} = \frac{1}{\sqrt{n-1}}. \tag{41}$$

In this case, the size of the first cone of silence is determined *solely by the number of elements*. On the other hand, if $\bar{\psi} = 2\pi - 2\pi/n$, the nulls are equispaced on the unit circle and we have an ordinary uniform array; then

$$1 - \cos\theta_1 = \frac{2\pi}{n\beta\ell} = \frac{\lambda}{n\ell}, \qquad \sin\frac{\theta_1}{2} = \sqrt{\frac{\lambda}{2n\ell}}. \tag{42}$$

This time the size of the first cone of silence depends upon the total length $L = (n-1)\ell$ of the array measured in wavelengths.

When the number of elements in the first case and the total length of the array in the second are large, then we have approximately

$$\theta_1' = \frac{2}{\sqrt{n-1}}, \qquad \theta_1'' = 2\sqrt{\frac{\lambda}{2n\ell}}. \tag{43}$$

For a large n the ratio of the two cone angles is approximately

$$\frac{\theta_1'}{\theta_1''} = \sqrt{\frac{2\ell}{\lambda}}. \tag{44}$$

For example, if $\ell = \lambda/8$, the angle of the major lobe in the first case is one-half of that in the second case or one-quarter if we are to compare the solid angles.

Equispacing the null points in the range of z not only makes the major lobe narrower but it also makes it sharper. Thus at the point lying halfway between the point of maximum radiation and the first null point, the field strength relative to the principal maximum is

$$\chi = \frac{\sin\dfrac{\bar{\psi}}{4(n-1)}\sin\dfrac{3\bar{\psi}}{4(n-1)}\sin\dfrac{5\bar{\psi}}{4(n-1)}\cdots\sin\dfrac{(2n-3)\bar{\psi}}{4(n-1)}}{\sin\dfrac{\bar{\psi}}{2(n-1)}\sin\dfrac{2\bar{\psi}}{2(n-1)}\sin\dfrac{3\bar{\psi}}{2(n-1)}\cdots\sin\dfrac{(n-1)\bar{\psi}}{2(n-1)}}. \tag{45}$$

For a quarter wavelength separation between the elements $\bar{\psi} = \pi$ and this ratio is equal to

$$\chi = \frac{1}{\sqrt{2(n-1)}} \tag{46}$$

so that the drop in the radiation intensity becomes $[10 \operatorname{Log}_{10} (n-1) + 3]$ decibels. On the other hand, for a long uniform array the corresponding drop is independent of n and is equal to 4 decibels.

Another consequence of equispacing the null points in the range of z consists in substantial suppression of subsidiary radiation lobes. The first subordinate maximum is situated approximately halfway between the first two null points where $\psi = -\frac{3\bar{\psi}}{2(n-1)}$; thus the field strength there, relative to the principal maximum, is

$$\chi = \frac{\sin \frac{\bar{\psi}}{4(n-1)} \sin \frac{\bar{\psi}}{4(n-1)} \sin \frac{3\bar{\psi}}{4(n-1)} \cdots \sin \frac{(2n-5)\bar{\psi}}{4(n-1)}}{\sin \frac{\bar{\psi}}{2(n-1)} \sin \frac{2\bar{\psi}}{2(n-1)} \sin \frac{3\bar{\psi}}{2(n-1)} \cdots \sin \frac{(n-1)\bar{\psi}}{2(n-1)}}. \tag{47}$$

For a quarter wavelength separation this field strength becomes

$$\begin{aligned} \chi &= \frac{2^{-\frac{2n-3}{2}} \sin \frac{\pi}{4(n-1)}}{\sqrt{2(n-1)}\, 2^{-\frac{2n-3}{2}} \sin \frac{(2n-3)\pi}{4(n-1)}} \\ &= \frac{\sin \frac{\pi}{4(n-1)}}{\sqrt{2(n-1)} \sin \frac{2(n-3)\pi}{4(n-1)}}. \end{aligned} \tag{48}$$

When n is sufficiently large, we have approximately

$$\chi = \frac{\pi}{4(n-1)\sqrt{2(n-1)}} \tag{49}$$

and the subsidiary maximum is $[30 \operatorname{Log}_{10} (n-1) + 5]$ decibels below the principal maximum. Each time the number of elements is doubled, the level of the subsidiary maximum is diminished by about 9 decibels. Thus an array of the type (37) with $\bar{\psi} = 2\beta\ell$ has very sharp directive properties.

In order to find the relative amplitudes and phase deviations of the elements of the array represented by (37), we expand $\sqrt{\Phi}$ into a single polynomial as follows[11]

$$\begin{aligned}
\sqrt{\Phi} &= |(1 - t^{-1}z)(1 - t^{-2}z) \cdots (1 - t^{-n+1}z)| \\
&= \left| 1 + \sum_{k=1}^{n-1} (-)^k \frac{(1 - t^{-n+1})(1 - t^{-n+2}) \cdots (1 - t^{-n+k})}{(1 - t^{-1})(1 - t^{-2}) \cdots (1 - t^{-k})} t^{-\frac{k(k+1)}{2}} z^k \right| \\
&= \left| 1 + \sum_{k=1}^{n-1} (-)^k \frac{(t^{-\frac{n-1}{2}} - t^{\frac{n-1}{2}}) \cdots (t^{-\frac{n-k}{2}} - t^{\frac{n-k}{2}})}{(t^{-\frac{1}{2}} - t^{\frac{1}{2}})(t^{-1} - t) \cdots (t^{-\frac{k}{2}} - t^{\frac{k}{2}})} t^{-\frac{kn}{2}} z^k \right| \qquad (50) \\
&= \left| 1 + \sum_{k=1}^{n-1} (-)^k \frac{\sin\frac{(n-1)\bar{\psi}}{2(n-1)} \cdots \sin\frac{(n-k)\bar{\psi}}{2(n-1)}}{\sin\frac{\bar{\psi}}{2(n-1)} \cdots \sin\frac{k\bar{\psi}}{2(n-1)}} e^{\frac{ikn\bar{\psi}}{2(n-1)}} z^k \right|.
\end{aligned}$$

Hence the progressive phase delay from one antenna to the next is equal to $\vartheta = \pi - \frac{n\bar{\psi}}{2(n-1)}$ and the amplitudes are in the ratio

$$\begin{aligned}
1, \quad & \frac{\sin\frac{(n-1)\bar{\psi}}{2(n-1)}}{\sin\frac{\bar{\psi}}{2(n-1)}}, \quad \frac{\sin\frac{(n-1)\bar{\psi}}{2(n-1)} \sin\frac{(n-2)\bar{\psi}}{2(n-1)}}{\sin\frac{\bar{\psi}}{2(n-1)} \sin\frac{2\bar{\psi}}{2(n-1)}}, \\
& \frac{\sin\frac{(n-1)\bar{\psi}}{2(n-1)} \sin\frac{(n-2)\bar{\psi}}{2(n-1)} \sin\frac{(n-3)\bar{\psi}}{2(n-1)}}{\sin\frac{\bar{\psi}}{2(n-1)} \sin\frac{2\bar{\psi}}{2(n-1)} \sin\frac{3\bar{\psi}}{2(n-1)}}, \quad \cdots, \quad 1.
\end{aligned} \qquad (51)$$

The amplitudes of the elements equidistant from the ends of the array are equal. In the special case of an end-on array with nulls equispaced in the range of z, $\bar{\psi} = 2\beta\ell$ and $\vartheta = \beta\ell$; hence the progressive phase delay from one antenna to the next is $\pi - \frac{\beta\ell}{n-1}$.

While (50) serves well for finding the amplitude and phase distribution in the individual elements of the array, another form is more general for calculating the directive properties. In order to obtain this form we set

[11] Chrystal's Algebra, Vol. 2, p. 340, (1926).

$$p_0 = 1, \qquad p_{n-1-k} = p_k, \qquad p_k = \frac{\sin \dfrac{(n-1)\bar{\psi}}{2(n-1)} \cdots \sin \dfrac{(n-k)\bar{\psi}}{2(n-1)}}{\sin \dfrac{\bar{\psi}}{2(n-1)} \cdots \sin \dfrac{k\bar{\psi}}{2(n-1)}} \tag{52}$$

$$\varphi = \psi + \frac{\bar{\psi}}{2} + \frac{\bar{\psi}}{2(n-1)} + \pi,$$

divide the last expression in (50) by $e^{\frac{i(n-1)\varphi}{2}}$ and combine the terms equidistant from the ends. Thus we obtain

$$\sqrt{\Phi} = 2 \cos \frac{(n-1)\varphi}{2} + 2p_1 \cos \frac{(n-3)\varphi}{2} + 2p_2 \cos \frac{(n-5)\varphi}{2} + \cdots, \tag{53}$$

where the last term is $2p_{\frac{n}{2}-1} \cos \frac{\varphi}{2}$ if n is even and $p_{\frac{n-1}{2}}$ if n is odd.

Let D be the maximum value of $\sqrt{\Phi}$; then the gain of the array over a single source is given by

$$G = 10 \operatorname{Log}_{10} \frac{4\pi D^2}{\iint \Phi \, d\Omega} = 10 \operatorname{Log}_{10} \frac{2D^2}{\int_0^\pi \Phi \sin \theta \, d\theta} \text{ decibels}, \tag{54}$$

where Ω is the solid angle and the integration is extended over a unit sphere. For an end-on array with nulls equispaced in the range of z, the maximum radiation is in the direction $\psi = 0$. Thus we shall have

$$D = 2 \cos \frac{(n-1)\varphi_0}{2} + 2p_1 \cos \frac{(n-3)\varphi_0}{2} + 2p_2 \cos \frac{(n-5)\varphi_0}{2} + \cdots, \tag{55}$$

where

$$\varphi_0 = \beta\ell + \frac{\beta\ell}{n-1} + \pi. \tag{56}$$

A convenient expression for the radiation intensity can be obtained from (50) by taking its norm

$$\Phi = [p_0 + p_1 e^{i\varphi} + p_2 e^{2i\varphi} + \cdots + p_{n-1} e^{i(n-1)\varphi}] \cdot [p_0 + p_1 e^{-i\varphi} + \cdots + p_{n-1} e^{-i(n-1)\varphi}]. \tag{57}$$

Since the set of coefficients $p_0, p_1, p_2 \cdots p_{n-1}$ is symmetric about the center, we find

$$\begin{aligned}\Phi = {} & 2p_0^2 \cos (n-1)\varphi + 2(p_0p_1 + p_1p_0) \cos (n-2)\varphi \\ & + 2(p_0p_2 + p_1p_1 + p_2p_0) \cos (n-3)\varphi \\ & + 2(p_0p_3 + p_1p_2 + p_2p_1 + p_3p_0) \cos (n-4)\varphi + \cdots \\ & + 2(p_0p_{n-2} + p_1p_{n-3} + p_2p_{n-4} + \cdots + p_{n-2}p_0) \cos \varphi \\ & + (p_0p_{n-1} + p_1p_{n-2} + p_2p_{n-3} + \cdots + p_{n-1}p_0).\end{aligned} \tag{58}$$

Since

$$\begin{gathered}\int_0^{\pi} \Phi \sin \theta \, d\theta = \frac{1}{\beta\ell} \int_{\varphi_1}^{\varphi_0} \Phi \, d\varphi, \\ \varphi_0 = \beta\ell - \vartheta + \frac{\bar{\psi}}{2} + \frac{\bar{\psi}}{2(n-1)} + \pi, \qquad \varphi_1 = \varphi_0 - 2\beta\ell\end{gathered} \tag{59}$$

we can write

$$\begin{aligned}\int_0^{\pi} \Phi \sin \theta \, d\theta = {} & \frac{1}{\beta\ell}\left[\frac{2p_0^2 \sin (n-1)\varphi}{n-1} + \frac{2(p_0p_1 + p_1p_0) \sin (n-2)\varphi}{n-2}\right. \\ & \left. + \cdots + (p_0p_{n-1} + p_1p_{n-2} + \cdots + p_{n-1}p_0)\varphi\right]_{\varphi_1}^{\varphi_0}.\end{aligned} \tag{60}$$

For an end-on array with nulls equispaced in the range of z, (60) becomes

$$\begin{aligned}\int_0^{\pi} \Phi \sin \theta \, d\theta = {} & \frac{2}{\beta\ell}\left[\frac{2(-)^{n-1}p_0^2}{n-1} \sin (n-1)\beta\ell \cos \frac{(n-1)\beta\ell}{n-1}\right. \\ & + \frac{2(-)^{n-2}(p_0p_1 + p_1p_0)}{n-2} \sin (n-2)\beta\ell \cos \frac{(n-2)\beta\ell}{n-1} + \cdots \\ & \left. + (p_0p_{n-1} + p_1p_{n-2} + \cdots + p_{n-1}p_0)\beta\ell\right].\end{aligned} \tag{61}$$

Substituting in (54), we shall obtain the gain of the array.

Similar expressions can be obtained for an end-on array in which the amplitudes of the individual elements are equal. Thus we have

$$\begin{aligned}\Phi = {} & \frac{1}{n^2}[e^{i(n-1)\psi} + e^{i(n-2)\psi} + \cdots + e^{i\psi} + 1] \\ & \cdot[1 + e^{-i\psi} + e^{-2i\psi} + \cdots + e^{-i(n-1)\psi}] \\ = {} & \frac{1}{n^2}[2 \cos (n-1)\psi + 4 \cos (n-2)\psi \\ & + 6 \cos (n-3)\psi + \cdots + 2(n-1) \cos \psi + n],\end{aligned} \tag{62}$$

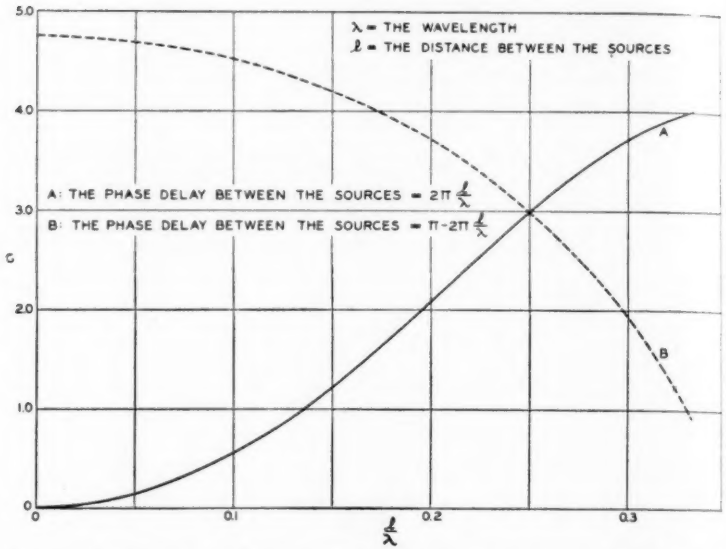

Fig. 21—The directive gain in decibels of a pair of sources with equal amplitudes. (A), the phase delay between the sources is $2\pi\ell/\lambda$; (B), the phase delay between the sources is $\pi - 2\pi\ell/\lambda$.

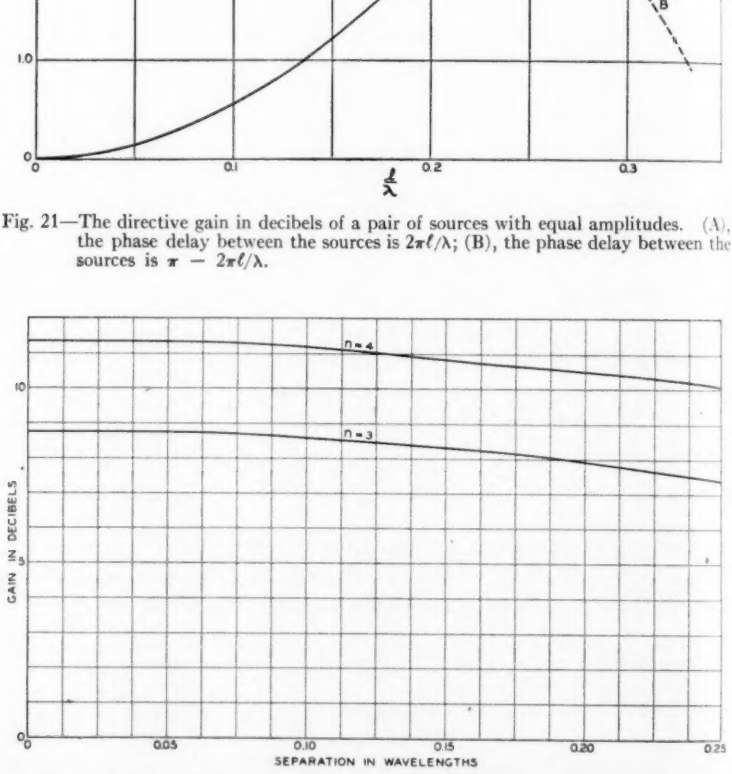

Fig. 22—The gain as a function of separation in wavelengths: n is the number of elements.

where $\psi = \beta\ell\,(\cos\theta - 1)$. In this case $D = 1$ and

$$\int_0^\pi \Phi \sin\theta\, d\theta = \frac{2}{n^2\beta\ell}\left[n\beta l + \frac{\sin 2(n-1)\beta\ell}{n-1} + \frac{2\sin 2(n-2)\beta\ell}{n-2} + \cdots + \frac{3\sin 2(n-3)\beta\ell}{n-3} + \cdots + (n-1)\sin 2\beta\ell\right]. \tag{63}$$

When the separation between the elements is exactly an integral number of quarter wavelengths, (63) becomes

$$\int_0^\pi \Phi \sin\theta\, d\theta = \frac{2}{n} \tag{64}$$

and consequently the gain is

$$G = 10\,\mathrm{Log}_{10}\, n. \tag{65}$$

Figure 21 contrasts the directive gain of a pair of sources of equal strength with the phase delay $2\pi\ell/\lambda$ (Curve A) with a directive gain of another pair of sources of equal strength but with the phase delay $\pi - 2\pi\ell/\lambda$ (Curve B). In one case the directive gain diminishes with separation between the elements and in the other it increases. Figure 22 shows the directive gain of three-element and four-element end-on arrays with nulls equispaced in the range of z.

As the separation between the elements decreases, the directive gain of an end-on array with nulls equispaced in the range of z increases but the radiation intensity per ampere-meter decreases. This circumstance would be of no importance if we had perfect conductors at our disposal to make transmitting and receiving antennas; but in reality parasitic losses in themselves cannot be removed and the efficiency of an array decreases, therefore, with the separation between the elements. This decrease in efficiency will impose an upper limit on the overall gain that can be obtained with small antenna arrays in spite of the fact that the directive gain could be made very large.

Likewise the band width diminishes as the distance between the elements decreases. This imposes another limitation on arrays of this type.

Memorial to the Classical Statistics

By KARL K. DARROW

ONE of the most elusive and perplexing, hazy and confusing of the parts of theoretical physics is that which bears the name of "statistical mechanics".* On the principle that a tree is to be judged by its fruits, this must be ranked as high as the tree which bore the golden apples of the Hesperides; for among its fruits are the Maxwell-Boltzmann distribution-law, the black-body radiation law, the value of the chemical constant, the Fermi distribution-law for the electrons in metals, the alternating intensities in band-spectra—and indeed the tree might lay a valid claim to the whole of quantum-theory. The singular thing is that such wonderful fruits should have grown from, or should have been grafted upon, so badly-rooted a tree. To change the metaphor, one frequently feels that the superstructure is sustaining the foundations, and the premises are flowing from the consequences, rather than the other way about. Perhaps anyone who feels this way should be disqualified from writing about the subject; but on the present occasion, the attempt is going to be made.

Statistical mechanics—hereinafter to be called "S.M." at times for short—did not of course arise from any desire to solve the problems suggested above, which came late. It seems to have sprung from attempts to answer older questions, of which the following may serve as an example. Consider a gas in a box, with an electric fan or something of the sort fitted inside to stir it up. The gas having been stirred up, the fan is stopped, leaving it in a state of surging and whirling about within the confines of the box. Very shortly, however, the surging and the whirling cease, the gas having passed of itself into a state of tranquillity and uniformity—uniform density, uniform pressure, uniform temperature. From this state it never departs, unless stirred up afresh. There is a tendency of the gas to go of itself from the state of surging into the state of uniformity, and no tendency at all for it to go from the state of uniformity back into the state of surging. This is very unlike the behavior of a pendulum, which having fallen from one end of the arc of its sweep to the middle thereof, moves on to the opposite end, retraces its path and returns to its first situation. Why should the gas behave that other way?

* I acknowledge with gratitude the incentive given me by Smith College to explore this subject, by offering me the opportunity of giving a course on statistical and chemical physics in the spring semester of 1942.

For an answer to this question and others of the kind, S.M. offers the following statement:

BASIC THEOREM OF STATISTICAL MECHANICS

A system is more likely to be found in a state of greater probability than in a state of lesser probability

It may be that no reader of these lines has ever seen the basic theorem of S.M. set forth with such merciless candor, though in many a sober treatise there is an elaborate statement which when analyzed turns out to be just this and nothing more. Of course it is a tautological statement, and has no value except insofar as it may help to drive some contradictory notion out of the student's mind or to prepare that mind for some meaning or other which is not yet in the statement but may be added to it later. Actually it can serve both these offices.

To be expelled from the mind of the student is first of all the idea that S.M. is going to give him a description of the way in which the gas proceeds from the surging state to the uniform. From an astronomer he may learn the orbit of the moon from apogee to perigee. From an authority on ballistics he may find the trajectory of the bullet from the muzzle of the gun to the bull's eye on the target. From a railroad office he may get a timetable showing the passage of the train from mile to mile over the rails from Boston to Chicago. All this sort of thing is out of the range of statistical mechanics! If a railroad acted like a surging gas and its timetable were devised in the spirit of S.M., one would go to the office and be told that the trains were enormously more likely to be in Boston than in Chicago or anywhere in between. From this one would be expected to infer that at any moment chosen at random the chance of finding a train anywhere along the line except in Boston would be practically nil—unless indeed one got a train and put it on the rails at Springfield, and even this would be of little use for getting to Chicago, since at every subsequent instant the train would almost certainly be in Boston. Not a very useful timetable, and not a very useful railroad!

S.M. thus starts off with a renunciation. It renounces the prospect of telling just how the gas proceeds from the surging state to the uniform state. To that smooth unbroken sequence of times and places whereby the moon finds its way through the heavens and the bullet through the air and the train along the rails, there is no counterpart presented.

This of course is a serious matter, for the smooth unbroken sequence is inseparably linked—or almost inseparably linked—with the notion of cause-and-effect, the notion of natural law, the notion of man as a being who can foretell the future. Mechanics harmonizes with these notions; for mech-

anics is the science which professes that, given the positions and momenta and the forces in a system of particles at 10 A.M. sharp, it can predict the positions and momenta of all the particles at 11 A.M. and every instant in between and all through the endless future. Statistical mechanics, for all the implications of its name, is nowhere nearly so audacious.

Suppose the electric fan, or whatever stirring-gadget was employed, was stopped an instant before 10 A.M. sharp. S.M. limits itself to affirming that at 11 A.M. the most probable state for the gas in the container—the immensely most probable state, the almost-certain state—is the uniform state. It also says the same thing exactly for 10:15 A.M., and for 10:01 A.M., and even for 10 A.M. sharp. If at 10 A.M. sharp the gas is in a state of wild and furious surging, S.M. does not deny the fact, but sees no reason for revising its own affirmation. If at 10:01 A.M. the gas is settling down but has not yet quite reached the uniform state, that again does not deter S.M. from standing by its assertion. Whenever a freak of chance or act of man may have produced one of the states which it calls improbable, S.M. just says "wait, and you shall have the state which *I* am going to talk about." To further questions it can only say "I know my limits"—and that is what its basic theorem says for it.

If now the negative aspect of the basic theorem is sufficiently clear, we may address ourselves to the task of giving the theorem a positive meaning. For this there is but one way: the word "probability" must be replaced with some word or phrase or mathematical expression which does have a meaning. After this is done we can of course restore the word "probability" as an equivalent for that other word or expression. The basic theorem will then be tautological upon the surface only, for actually it will have the meaning conferred upon it by the definition of its key-word.

Various meanings have been offered for the key-word, by various people who have been successful in getting useful results out of statistical mechanics. Until 1924 the dominant meaning was that imposed by Gibbs and Boltzmann. From this meaning arises the form of S.M. which is called "the classical statistics". (The word "statistics", by the way, is a bad but common abbreviation for "statistical mechanics".) This is the topic of the present article. In 1924 there was proposed a novel meaning for the key-word, which led to results sometimes agreeing with, sometimes differing from, those attained by the classical statistics. Where the results of the two agreed, they agreed with experiment also; where the results of the two disagreed, experiment sustained the new one. This event has left the classical statistics in a strange situation, in which one cannot exclude the possibility that all of its remarkable achievement is due to a happy but deceptive chance. The classical statistics may indeed be only a past episode in the history of scientific thought, and it is for this reason that I have given to the article the

strange and sombre title "Memorial to the Classical Statistics". Yet even as a past episode, it is worthy of remembrance; its didactic value may yet be great; and perhaps the human mind may some day stretch its powers to the point of conceiving the classical and the new statistics as aspects of a single whole, as it has lately stretched itself to the extent of uniting the wave-picture and the corpuscular picture of matter and of light.

The Maxwell Statistics

Since the main concern of S.M. is with the "most probable state", one sees that its principal content must be made up of assertions about that most probable state. Maxwell made such an assertion. He wrote down a formula for the distribution-in-velocity of the molecules of a gas. It is the formula now called "the Maxwell-Boltzmann distribution-law", which is so well known to the readers of this journal that I will not bother to write it down until there is actual need for having it on the page. Maxwell might have said bluntly: "This is the distribution which I will assume for the most probable state"; and having said so, left it at that. He did not leave it at that, and presumably he would have been dissatisfied so to leave it, as most of us would be. Instead, he postulated a pair of attributes for the most probable state, and showed that if these are the attributes, then the distribution is according to that formula.

The attributes which Maxwell postulated are "isotropy" and "independence".

The former is easy enough. One assumes that in the most probable state, the distribution of velocities of the molecules is isotropic. Nothing can usefully be added to this simple statement.

The latter is a little harder to grasp. Perhaps it can best be exhibited by describing a couple of imagined cases for which it would not be valid. Suppose for instance that all of the molecules have the same speed—the same magnitude of velocity, though their velocity-vectors be pointed in all directions. Let this common value of speed be denoted by V, and let any direction chosen at random be made the axis of x in an ordinary coordinate-frame. If a molecule happened to be travelling with such a velocity that the component thereof along the x-axis, v_x let us call it, was just equal to V, then it would be a certainty that v_y and v_z, the y and z components of the velocity, were both of them zero. If a molecule happened to be travelling in such a way that v_x was zero, then either v_y or v_z or both of them would have to be different from zero, and the square root of the sum of the squares of v_y and v_z would have to be equal to V. There would consequently be a *correlation* between the values of the three components, and the probable—nay even the possible—values of any one of them would be affected by those of the other two. If the molecules had a uniform distribution of speeds up

to a maximum value of V, there would still be a correlation of a similar sort, though not so marked a one: the higher the velocity-component in the x-direction, the lower would the y and the z components be likely to be. One could imagine distributions for which the higher the velocity-component in the x-direction, the higher would the y and the z components be likely to be.

The "assumption of independence" is, that in the most probable state there is no correlation at all. Whether the x-component of the velocity of a molecule is high or low is a detail which has no influence whatever on the possible or the probable values of the y and the z components. Low values of v_y go just as well and just as abundantly with low values of v_x as with high, and reversely.

The Maxwell-Boltzmann law, as I said, is the distribution-law which conforms to both the assumption of isotropy and the assumption of independence. So the question arises: do those two assumptions have the quality of plausibility and of convincingness, which make the average person say "Surely these must be the attributes of the most probable state of a gas!" I do not know what result a referendum on this question would give, but it is my guess that most physicists would feel more satisfied with these than they would with the Maxwell-Boltzmann distribution-law if it were tossed out to them with the bare affirmation "This is assumed to be the attribute of the most probable state". Clearly this is how Maxwell felt, and there is no better guide than the intuition of a Maxwell.

The foregoing question is something else than the question whether the assumptions, and the Maxwell-Boltzmann distribution-law which follows from them, are truly the attributes of the most probable state. It is a strange historical fact that not for many years after the promulgation of this famous law, and not till after both of its sponsors were dead, was there any proper test of it. The derivations of the law were exercises in abstract and unrenumerated thought. Nevertheless experiment—applied to thermionic electrons, to molecules of ordinary gases, to thermal neutrons—came at long last to justify Maxwell. To any who may feel that the assumption of independence is in itself too reasonable to require any proof, I disclose that in other forms of statistics this assumption is declared to be false, except as an approximation.

The "Maxwell statistics" therefore consists in the main of the statement: *The most probable state of a gas is that in which isotropy and independence prevail among the velocity-vectors of the molecules.*

We now require some terminology and some notation.

I take for granted an understanding of the terms "velocity-vector" and "distribution-in-velocity", these being learned by physicists out of kinetic theory if not out of S.M. A velocity-vector may be replaced by a point

which serves just as well for all of its purposes and even better for some. Let the velocity-components v_x, v_y, and v_z be laid out along the axes of a Cartesian coordinate-frame, and the vector for any molecule be drawn from the origin: the point at its tip is the point in question. Point and coordinate-frame are said to be "in velocity-space". Statistical mechanics prefers as a rule to deal with the momenta of the molecules rather than their velocities. This is for valid and powerful reasons, one of which is that the transition to the case of photons becomes much easier.[1] In the case of material gases it makes no practical difference, since the momentum of a molecule is its velocity-vector multiplied by the mass of the molecule which is practically a constant, and every statement about the distribution-in-velocity can with the utmost ease be translated into a statement about the distribution-in-momentum and *vice versa*. The momentum-vector may be replaced by the point at its tip, having coordinates p_x, p_y and p_z in a coordinate-frame in "momentum-space". If we consider together with these the three coordinates x, y, z of the molecule in ordinary space, we may say that we are locating the molecule in six-dimensional space. I have yet to meet someone who claims that he can visualize a six-dimensional space, and yet there is no doubt that the phrase fulfills a psychological need and has a practical value. The six-dimensional space of these particular six variables is called "the μ-space".

It seems odd to bring in the μ-space before considering by itself the three-dimensional "ordinary" or "coordinate-space" in which the gas is located. Is there nothing to be said about the most probable distribution of the molecules in the coordinate-space? Well, "every schoolboy knows" that the state to which a gas tends and in which it remains is a state of uniform density. Maxwell, I think, accepted this as one of the facts behind which one cannot, or does not, go. For a complete statement of the Maxwell statistics I therefore offer the following:

A gas is very much more likely to be in its "most probable state" than in any other. The most probable state is that in which isotropy and independence prevail among the momentum-vectors, while the distribution in coordinate-space is uniform.

So in the Maxwell statistics the distribution-in-momentum of the molecules is *derived* from assumptions ostensibly more basic, while the distribution-in-ordinary-space is simply *affirmed*. If a theory could be devised in which both were derived from assumptions apparently more basic, one would be likely to feel that something had been gained. Now this is a characteristic, and one of the principal virtues, of Boltzmann's theory known as the "Boltzmann Statistics" or as the "Classical Statistics".

[1] Another reason has to do with "Liouville's theorem," for which unfortunately I cannot make room without overloading this article.

The Boltzmann Statistics

Boltzmann invented a way of appraising the probability of any imagined state of a gas, which has the following very remarkable features:

(*a*) It gives so sharp a definition to the key-word "probability", that not only can the state of maximum probability be identified, but the ratio of the probabilities of any two states can be computed.

(*b*) For the distribution-in-momentum of the molecules in the most probable state, it derives a formula identical with that which springs from the Maxwell statistics. This of course is why the formula is known as the Maxwell-Boltzmann law.

(*c*) For the distribution-in-space of the molecules in the most probable state, it derives the uniform distribution.

All this does not entail that the Boltzmann statistics is necessarily right. It does, however, lead to consequences, which it is the privilege and the affair of experimental physics to verify or to reject.

I can now write down a phrase into which the Boltzmann statistics, and equally well those which came later, can be fitted:

The probability of a state is the number of different ways in which the state can be realized.

This is another of those oracular sayings which acquire a meaning only after some meaning is given to the key-word, which is this case is *ways*. I could now rewrite the basic theorem without the word "probability", and so can the reader; but the only effect would be to transfer the mystery out of the word "probability" and into the word "way". Boltzmann, however, assigned a meaning to the latter word. It is this meaning which we now must strive to realize.

For this purpose I propose a game of which the outfit consists of a sack, an enormous number N of balls, and a smaller number M of baskets. The game is played by reaching into the sack, drawing out the balls one after another, and tossing them into the baskets. All of the balls feel precisely alike to the hand, so that there is never the least inclination to put one aside and pick up another as one's hand gropes around in the sack. Nevertheless when one looks at the balls after they have fallen into the baskets, one sees that they are nicely adorned with the integer numbers running from 1 to N. Incidentally the baskets also are numbered. It is this numbering which gives point to the game.

Someone or other—someone who might be designated as the caller, after the man who calls the figures of a square-dance—has prescribed a sequence of M numbers N_1 and N_2 and N_3 and so on to N_M, all of them positive integers and totalling up to N. A single inning of the game consists in drawing all of the balls out of the sack one after another, and dropping the first N_1 which come out into the basket I, the next N_2 which emerge into the

basket II, and so on until every one of the balls is reposing in one or another basket. Now along comes the umpire with pencil and pen, and he writes down on one sheet of his pad of paper the numbers of all the balls which are in basket I, and on a second sheet the numbers of all which are in II, and so on until he has got an inventory of the contents of all of the baskets. *The inventory does not state the order in which the balls in any basket were dropped into that basket.* That order is blotted out and forgotten. *The inventory states which balls are in which baskets*, and lets it go at that.

This does not seem a very entertaining game, but entertainment is not what it is for. The present question is: how many different inventories can there be, consistent with that sequence of figures $N_1, N_2, N_3, \cdots N_M$ which the caller prescribed at the start?

The answer is obtained in what must seem, to anyone meeting for the first time such a question, a strangely devious way.

First we evaluate the whole number of different *orders* in which the balls can be drawn from the sack. This is N-factorial or $N!$; for the first ball to emerge may be any one of the N, and the next may then be any one of the $(N-1)$ remaining, and the next may then be any one of the $(N-2)$ remaining, and so on to the end.

If each order corresponded to a different inventory, $N!$ would be our answer. Clearly this is so, if and only if there are as many baskets as balls and one ball in every basket. In all other cases $N!$ is larger, and often colossally larger, than the number which we seek. It is necessary now to see that this great multitude of $N!$ different orders falls into groups composed of X orders apiece, all of those in a single group corresponding to a single inventory—necessary to see this, and to calculate X; whereupon we shall find that X, the "number of orders per inventory", is the same for all of the inventories—so that the number which we seek is $N!$ divided by this common value of X.

It seems to be helpful to think of some one inventory, and of some one order which leads to that inventory. By a certain amount of mental effort, which varies from person to person, it can be seen that this particular order is but one among $N_1!\, N_2!\, N_3! \cdots N_M!$ different orders all leading to the very same inventory. For think of the N_1 numbered balls which lie in the first of the baskets: there are $N_1!$ different orders in which they could have come out of the sack, and every one of these corresponds to the very same inventory. Think next of the N_2 numbered balls which rest in the second basket: they might have come out of the sack in $N_2!$ different orders, without changing the inventory. Think now of the contents of both of these baskets at once. Each of the $N_2!$ orders in which the second basketful may come out of the sack may follow on any one of the $N_1!$ orders in which it is possible for the first basketful to emerge. The product $N_1!\, N_2!$ is therefore the total

number of ways in which the first $(N_1 + N_2)$ of the balls might have come out of the sack without changing the inventory.

The process of proof need not be carried further. X has been evaluated. It bears no earmark of whatever particular inventory the student may have chosen to adopt at the beginning. It depends only upon the sequence of numbers $N_1, N_2, \cdots, N_M$ fixed by the caller, which sequence I will hereafter term a "distribution".

The number of inventories—or "complexions", to use a commoner word—for the distribution $N_1, \cdots, N_M$ is therefore given by the formula,

$$W = N!/N_1!\, N_2! \cdots N_M! = N!/\Pi N_i! \tag{1}$$

The theorem to which we are advancing affirms that this number has its maximum value for the uniform distribution—the distribution in which the caller assigns the same number of balls, N/M, to each of the baskets.

The usual argument for this statement may be put as follows: Let us assume the uniform distribution, with $A = (N/M)$ balls in each basket, and compare its value of W with that of one of the neighboring distributions such as the one in which there are $(A + 1)$ balls in the first of the baskets, $(A - 1)$ in the second and A in each of the rest. It is not even necessary to get out a pencil and paper to see that W for the latter is less than W for the former, being in fact just $A/(A + 1)$ times as great. The same is evidently true for disarrangements of the uniform distribution which involve more than two baskets and more than one ball per basket. The conclusion is clinched by the obvious fact that when all of the balls are in any one basket, W has its least possible value, *viz.* unity. (To unite this formally with the previous statements, one must follow the mathematicians' practice of using a symbol 0! or "zero-factorial" and giving it the value unity).

We shall have to play this not so very entertaining game on several occasions in S.M., altering the meaning of the balls and the meaning of the baskets from one occasion to the next. The reader has probably guessed that the balls stand for the molecules. The guess is right in the classical statistics, wrong in the newer forms. To get at the meaning of the baskets, suppose the gas contained in a box of volume V, the interior of which is divided up by impalpable coordinate-planes into compartments or cells all of the same volume V_0. The baskets stand for the cells.

Now we have the theorem that W is greatest for the uniform distribution of the balls in the baskets, and the assertion that the most probable state of a gas is the state of uniform density, all ready to be fitted together. The process of fitting-together is of the simplest. W is christened the "probability" of the state described by the "distribution" $N_1, N_2, \cdots N_M$, the quantities N_i now standing for the numbers of molecules in the various cells. Not only is the state of uniformity the most probable one by this definition,

but *so long as the number of molecules N is many times as great as the number of compartments M*—a condition easy to realize—those distributions which are markedly far from uniform have probabilities which are fantastically smaller than the value of W for the uniform state.

The Boltzmann statistics manages thus to derive the assertion aforesaid—the assertion that the uniform distribution in ordinary space is of all the most probable—from a principle which (at least in appearance) is more fundamental. It has indeed a couple of bothersome points—more than a couple perhaps, but there are two in particular which the newer statistics will attempt to assuage. One of these is the size to be assigned to the cells V_0; but we are borrowing trouble to think too much of that now, since whatever choice be made *so long as N/M be large* will not affect the achievement just cited. The other is, that one would much rather think of the molecules of a gas (of a single chemical kind) as being alike absolutely, than as being distinguished one from another by a mysterious something-or-other represented in this theory by numbers painted on balls. In the Boltzmann statistics, however, the numbers must stay on the balls.

We go over into the momentum-space, setting up a coordinate-frame and representing the molecules by dots, the coordinates of which are the momentum-components p_x, p_y, p_z of the molecules in question. To each position of a dot corresponds an energy-value, equal to $(1/2m)\ (p_x^2 + p_y^2 + p_z^2)$; we will call it E. E vanishes at the origin, and has a constant value over any spherical surface centered at the origin. To any distribution of the dots will correspond a specific value for the total energy of the gas. For this we need a symbol different from E; and as we shall have a good deal to do with thermodynamics later on, I choose the thermodynamical symbol U. The average energy of the molecules of the gas will then be U/N, to be denoted by $\overline{U}$.

The entry of E and U into the situation is of the first importance. It is in fact all that will save us from the highly unwanted conclusion that the most probable distribution in the momentum-space is the uniform one, just as it was in the coordinate-space. To see why it makes so great a difference is not altogether easy. I think that the reflections which follow may give an inkling of the reason.

The momentum-space must be taken either as infinite or as finite. If we take it as infinite and demand a distribution of uniform density, then the density goes to zero and at the same time the energies of the molecules go to infinity, producing an impossible situation. Let us then take it as finite, blocking off all of the parts of it which lie beyond a certain sphere centered at the origin. Assume a uniform distribution within the sphere. This will correspond to a certain value of U. (The student may suppose, if it makes him happier, that the U-value was preassigned and the radius of the "certain sphere" chosen accordingly.) The W-value of this distribution will surely

be greater than that for any non-uniform distribution, whether of the same U or of a different U, confined within the sphere. However, by blocking off the whole of the momentum-space beyond the sphere, we have barred a whole lot of distributions corresponding to the same U and having some of their dots beyond the sphere. By no means have we proved that the W-value for the uniform distribution within the sphere is greater than that for any and all of the barred distributions. Now if we can agree that the blocking-off of part of the momentum-space is a silly thing to do and unacceptable to Nature, the argument for the uniform distribution is spoiled, and we have to look for a new idea.

At this point it seems best to go through the mathematical process for finding the distribution of greatest W in the coordinate-space and the momentum-space, just as that process is presented in the textbooks.

We return to equation (1) and make it a manageable one by having recourse to that godsend of statistical mechanics, the "Stirling approximation", which may be written thus:

$$\ln N! = N \ln N - N + \ln \sqrt{2\pi N} \tag{2}$$

This is valid only for large values of N, though writers on S.M. never seem to remember how large the values must be. For still larger values of N we can drop off the last two terms, arriving at a sort of super-Stirling approximation which however itself is commonly called *the* Stirling approximation:

$$\ln N! = N \ln N \tag{3}$$

Putting (3) into (1), we find:

$$\ln W = N \ln N - \Sigma N_i \ln N_i \tag{4}$$

Defining some quantities w_i by the equations $N_i = Nw_i$, we make this over into:

$$\ln W = -N \Sigma w_i \ln w_i \tag{5}$$

having availed ourselves of the obvious fact that Σw_i is equal to unity.

We might now convert this into an equation for W, but this would be a waste of time and energy, since whenever W has a maximum so also will $\ln W$. With $\ln W$, therefore, we operate from now on. Making small variations in the quantities N_i, and making therefore small variations—call them δw_i—in the quantities w_i, we find in first approximation for the ensuing change in $\ln W$,

$$\delta \ln W = -N \Sigma (1 + \ln w_i) \delta w_i \tag{6}$$

Now we are restricting ourselves to variations in the quantities N_i which leave unchanged the total number of molecules in the cells, or of balls in the baskets—to variations, therefore, for which

$$\Sigma N_i = N = \text{constant}, \qquad \Sigma \delta w_i = 0 \tag{7}$$

This restriction being introduced into (6), $\delta \ln W$ *proceeds to vanish if and only if* w_i *has the same value for all of the cells.* Now, the vanishment of $\delta \ln W$ is a necessary condition for having a maximum of W at the situation in question. I do not refer to it as a sufficient condition, because it admits of a minimum or of what is technically known as a "stationary" value of W in the situation in question. However it has already been shown, without the aid of the Stirling approximation, that the expression to which we are approximating is greater for the uniform distribution than for the neighboring non-uniform ones. It may therefore be accepted that here we have a maximum of W for the uniform distribution, and have reached the old result in a new way; an achievement nearly useless, were it not a prelude to the performance in momentum-space.

I continue to use the symbols W and N and N_i and w_i, but now with reference to the distribution of the representative dots in momentum-space. A new symbol, E_i, shall signify the energy of a molecule in the ith cell of the momentum-space. We wish at all costs to avoid the conclusion that the stable distribution in the momentum-space is the uniform one. Boltzmann managed to avoid it, and his was the following way:

Let us write, for the number of molecules in the ith cell, the expression:

$$Nw_i = NA \exp(-BE_i) \tag{8}$$

and insert it into (6). We shall find:

$$\delta \ln W = -N \Sigma (1 + \ln A - BE_i)\, \delta w_i . \tag{9}$$

Of the three terms on the right, two vanish for all variations in which the total number of molecules remains the same. The third does not—but it *will* vanish for a restricted class of these variations, to wit, those and those only for which the total energy of all the molecules remains the same; for $N\Sigma w_i E_i$ is precisely that total energy.

Some writers at this point ask the student to imagine a gas in a container being completely cut off from energy-interchange with the container-walls and with the whole of the outside world, and therefore being limited to the particular U-value with which it started out. Others import the word "temperature" which I am desperately (and vainly) trying to keep out until I am ready to bring it formally into the discourse, and aver that the gas is nearly or quite so limited if the walls of the container have the same temperature as the gas itself. The student may take his choice, but must suppose that

under such conditions Nature rejects that distribution which so to speak is "stable against" every conceivable variation, and elects that peculiar distribution which is stable not against any conceivable variation but only against the possible ones. Perhaps this is because the uniform distribution would entail the consequences mentioned on page 117, or perhaps there is no sense in saying that it is "because" of this or "because" of that. Anyhow, the peculiar distribution is the one which the data sustain.

However I have not really defined the peculiar distribution as yet, having merely thrown the symbols A and B into equation (8) as though they stood for completely disposable constants. It can readily be seen that at the most there can be but one disposable constant, for A and B interlinked by the obvious equation:

$$\Sigma w_i = A \Sigma \exp(-BE_i) = 1 \tag{10}$$

But even B is not disposable, if the total energy U and the average energy per molecule $\bar{U}$ are preassigned; for there is another obvious equation:

$$\bar{U} = \Sigma E_i w_i = A \Sigma E_i \exp(-BE_i) \tag{11}$$

What with equations (10) and (11), there is no longer anything disposable about the constants A and B. The peculiar distribution in the momentum-space is completely defined. It is the Maxwell-Boltzmann distribution-law obtained from the Maxwell statistics, and sometimes known as the "canonical" distribution.

To summarize now the Boltzmann statistics as on page 113 the Maxwell statistics was summarized:

A gas is more likely to be found in its most probable state than in any other. The probability of a state is found by imagining it as a distribution of numbered molecules among cells, in the coordinate-space and in the momentum-space. That of any distribution is measured by the number of inventories compatible therewith. By this criterion the most probable distribution in coordinate-space is the uniform one, and by this criterion carefully hedged about, the most probable distribution in momentum-space is the Maxwell-Boltzmann or canonical one. It is necessary to liken molecules of a single kind to numbered balls, differing in no way except the numbering.

This point was reached by statistical mechanics about fifty years ago. Had it not been for Planck's wish and tenacious will to explain the black-body radiation-law, it might have been the stopping-point.

A Helpful and Troublesome Coincidence between Two Different Quantities

Let us return to the game with the sack, the balls and the basket, played in the manner which led to good results when applied to the molecules in the coordinate-space.

The most probable distribution is the one evoked by the caller, when he calls for an equal number of balls in every basket. If there are N balls and M baskets, this means N/M balls to each basket, and a maximum number of inventories which I will call W_{max}. Looking back to equation (1), we see that W_{max} is a fraction the numerator of which is N-factorial, while the denominator is (N/M)-factorial raised to the power M. Taking logarithms and using the super-Stirling approximation, we find:

$$\ln W_{max} = N \ln M \tag{12}$$

The logarithm of the probability of the most probable distribution (of numbered balls in numbered baskets, or molecules in equal cells of coordinate-space) is equal to the logarithm of the number of baskets (cells), multiplied by the number of balls (molecules).

Next suppose the caller, in a fit of uncontrollable zest for the game, calling in succession every one of the conceivable distributions. What is the total number of inventories compatible with all of them together? To sum over every conceivable expression of the type of (1) seems a hopeless assignment, but there is a short-cut to the result.

Fix a particular order for the drawing of the balls from the sacks—it may as well be the very order of their numbering. The first of the balls to be drawn may be tossed into any one of the baskets, giving M distinct "possibilities". The second may be tossed into any one of the baskets, the same or another, giving in conjunction with the fate of the first M^2 different possibilities. The third may be tossed—but we leap to the conclusion. There are M^N possibilities altogether, and *these are the inventories*. Thus the total number of inventories consistent with all of the distributions, which I will call W_{tot}, is a number whereof the logarithm is,

$$\ln W_{tot} = N \ln M \tag{13}$$

But this is the same as the expression for *ln* W_{max}!

The meaning of this strange coincidence can only be, that when N and N/M are both so great that the super-Stirling approximation is a good one, then the ***logarithm*** of the number of inventories belonging to the most probable distribution is nearly as great as the *logarithm* of the total number of inventories belonging to all of the distributions put together—so nearly as great, that either *logarithm* is a good approximation to the other.

In the foregoing very important paragraph, I have italicized the word "logarithm" because if it were left out the statement would become a false one. The statement is not true if applied to the numbers themselves. W_{tot} is manyfold greater than W_{max}, and the ratio between the two actually increases with rising N. So does the difference between $\ln W_{tot}$ and $\ln W_{max}$ increase with rising N, but not so fast as either by itself; wherefore the truth

of the statement. The student may convince himself of this by applying the second-degree Stirling approximation (equation 2)[2].

I have called this both a helpful and a troublesome coincidence. It may be deemed a helpful one, because the expression for the total number of inventories is easier to derive and easier to remember than the expression for the number of inventories belonging to the most probable distribution. If therefore one has good ground for believing (as here is the case) that the logarithms of the two are approximately equal, one may serenely remember and use $\ln W_{tot}$ instead of $\ln W_{max}$. The troublesome feature is, that some expositors speak of $\ln W_{tot}$ throughout and never allude to $\ln W_{max}$, thus confusing the student to an extent which (if my experience is typical) may well be serious. I shall later dwell on the fact that $\ln W$ for any distribution is regarded as a measure of the entropy of that distribution, and $\ln W_{max}$ therefore as a measure of the entropy of the most probable distribution. Some people imply that $\ln W_{tot}$ is the true measure of the entropy of the gas, instead of being an approximation to it. They commit no numerical error in so doing, but they blot out the most remarkable quality of the Boltzmann statistics, to wit, the clear distinction which it makes between the most probable distribution and those of lesser probability. This mistake is more commonly made in treating the newer statistics. Here I am not so sure that it is a mistake, but I think so.

Meanings of the Word "State"

The word "state", which turns up continually in this essay, is one of those words of which a proper definition is hardly less than a full description of the theory which employs it. When the theory changes so also does the meaning of the word. In the welter of statistical theories, the word "state" has several different meanings. In thermodynamics also it has more than one meaning, but one is preeminent.

Thermodynamics usually concerns itself with gases (not to speak of liquids and solids) which are in what I earlier called a "uniform" state: uniform density, uniform pressure, uniform temperature. For a gas of a single kind ("kind" being a word which it is the business of chemistry to define) it is a fact of experience that any two of these three variables suffice to define the third and also all of the other variables which thermodynamics cares about. Of these others there are two in particular which I mention at this point, the

[2] Actually if one goes from the "most probable state" $N_i = const. = N/M$ to the "next most probable" in which one ball is taken out of one of the baskets and put into another, the change in W is in the ratio of (N/M) to $(N/M) + 1$, which is practically no change at all when N/M is so high as is commonly taken. This shows that the statement could not be true if it were made about the numbers W_{max} and W_{tot} rather than about the logarithms thereof. It certainly looks as though the statement could not be true even when made of the logarithms, but this is evidently one of the cases where "intuition" is a fallible guide.

energy and the entropy. This makes five altogether, and any two of the five suffice to determine the state,—THE STATE, the uniform state, the only one about which thermodynamics really knows or cares. When asked about what I earlier called "a surging state", thermodynamics mutters something to the effect that the entropy of such a state is smaller than that of THE STATE, and then puts an end to the conversation by refusing to commit itself further. Thermodynamics takes no cognizance of the molecular structure of matter. A gas might be a continuum, for all that it knows or cares.

Statistical mechanics talks about a mental image of the gas, in the form of a flock of dots in the coordinate-space and another flock of dots in the momentum-space, or one may call them a single flock of dots in the μ-space. In Boltzmann statistics, the "state" of this image is what I have been calling the "distribution". The most probable state of the image—to wit, the one with the greatest number of inventories or complexions—is identified with THE STATE of thermodynamics. All of the rest belong to the category of which thermodynamics would say, that the entropy is smaller than it is for THE STATE. But since according to S.M. they belong to a category for which the probability is smaller than it is for THE STATE, one sees a connection between entropy of the gas and probability of the image beginning to take shape.

Now it is time to make a formal introduction of the concepts of entropy and temperature—the latter word having already sneaked into this article two or three times in spite of all my efforts to keep it out.

Formal Entrance of Entropy and Temperature

For a substance, meaning now a gas, of a single kind, entropy and temperature are defined by the equation,

$$dU = TdS - PdV \tag{14}$$

P stands for pressure, V for volume, and S for entropy. For energy I use the symbol U already employed in that sense—but notice that formerly it stood for the kinetic energy of the molecules! To use the same symbol in both senses implies that the energy of the gas is entirely the kinetic energy (of translatory motion) of the molecules. This identification turns out to be valid for the "monatomic" gases, which are luckily numerous and well-studied. To these we confine ourselves throughout this article. T stands for the temperature called absolute; this being the only kind of temperature which will ever figure in this article, the adjective henceforth is discarded. Density was the fifth variable in my list given above, but volume is usually preferred to it. To make them equally useful, the quantity of gas must be stated; here it will be taken as one gramme-molecule.

It is evident that the equation is a comparison between two states. Do not go astray by supposing that these are like two of the states which we have been considering, having the same U and V and differing in the number of inventories! These on the contrary are two examples of THE STATE—of the thermodynamic state, of the most probable state—of a gas, differing in the values of some at least among the five variables. The quantities dU and dS and dV are the differences between the U-values and the S-values and the V-values of the two states, while P and T may be taken as referring to either, the smallness of the difference between the two states—implied by the differential notation—permitting of this.

It is also evident from my wording that the one equation is being used to define the two quantities S and T. This is unluckily no verbal slip, nor is it a temporary shortcut to be replaced by a royal road as the argument proceeds. The meanings of entropy and temperature are so coiled up together in thermodynamics, that it is impossible to take them apart unmutilated. One cannot seize either by storm and then invest the other, at least not without the aid of statistical theory: one has to surround them both in a single campaign. As Eddington has vigorously written, this is a common thing in physics. Electric force is defined as that which acts on electric charge, electric charge as that which is acted upon by electric force, and so on. . . . Common as it may be, it is probably nowhere else so harassing as in thermodynamics. There are three ways of intruding upon the vicious circle.

First, to apprehend both concepts in a single mental act. This is the counsel of perfection.

Second, to use a temporary definition of temperature, with the promise of confirming or correcting it later. The ideal-gas thermometer is the device used for this purpose in thermodynamics. Anyone trained in this way is likely to think for the rest of his life of temperature as the primary concept, entropy as a derived one—as indeed was the case, when thermodynamics started.

Third, to produce a theory which makes a pronouncement as to the nature of entropy.

This last is the major office of statistical mechanics. To those who accept it, entropy becomes the primary concept and temperature the derived one, and both are visualized by the aid of the key-word "probability" of the basic theorem, interpreted in some particular way.

Old Statistical Theory of Entropy

In the classical statistics, the entropy of a distribution is considered to be the logarithm of the number of inventories or complexions compatible with that dis-

tribution, multiplied by a constant (always denoted by k) which is adjusted to bring about agreement with experiment:

$$\textit{entropy } S = k \ln W. \tag{15}$$

To illustrate this doctrine and to evaluate k, I now take the student back to the coor linate-space, where a box of volume V populated with N molecules is ... led mentally into M equal cells of volume V_0, and the most probable distribution is characterized by the value $N \ln M$ for the logarithm of the probability. The entropy—or no, not the entire entropy of the gas, but merely what I will call "the contribution of the volume of the entropy" and denote by S_v—is then supposed to be $kN \ln M$, or:

$$S_v = kN \ln V - kN \ln V_0 \tag{16}$$

Reverting to the equation (14) in which the definitions of entropy and temperature were tangled up together, and rearranging it, we get:

$$TdS = dU + PdV \tag{17}$$

Now, an "ideal gas" is defined by tw ... ttributes. First, there exists between its pressure and its volume and its ... perature the relation $P = aT/V$, wherein a stands for a constant. Sec ..., its energy U depends upon the temperature only, and not upon any ot ... variable, in particular not upon the volume. Therefore we may write:

$$TdS = C_v dT + (aT/V)dV \tag{18}$$

C_v here standing for something of which we need only know that it is a function of T alone. Integrating, we find:

$$S = R \ln V + (\textit{function of temperature}) + \text{constant} \tag{19}$$

and lo! it is seen that the dependence of entropy on volume is precisely of the sort which the theory is fitted to explain.

The next step is to adjust the value of the constant k. The constant a aforesaid is proportional to the amount of gas in the box, proportional therefore to N: it is the constant ratio of a to N to which k must be equated. For the amount of gas let us choose one gramme-molecule. Then a assumes the value always symbolized by R and called the "gas-constant", and N assumes the value usually symbolized by N_0 and called the "Avogadro number". Both of these are known from experiment, and k is fixed by the equation

$$k = R/N_0 \tag{20}$$

The constant k is named in Boltzmann's honor, though in his time its value was not known because the value of N_0 was only vaguely apprehended.

Now we have settled what I called "the contribution of volume to en-

tropy". It remains to interpret the rest of the right-hand member of (19), which I will call "the contribution of temperature to entropy". To do this we must re-enter the momentum-space.

From (15) and (5) and (8) we get, for the entropy S_m of the flock of dots in the momentum-space:

$$S_m = -k \ln W = -kN \Sigma w_i \ln w_i$$
$$= -kNA \Sigma (\ln A - BE_i) e^{-BE_i} \tag{21}$$

Refreshing our memory from (10), we see that the first term of this expression reduces to $-kN \ln A$. Refreshing our memory from (11), we see that the second term reduces to $+ kNB\bar{U}$ or kBU. Referring now to one gramme-molecule of gas, I put R for Nk, and find:

$$S_m = -R \ln A + kBU \tag{22}$$

S_m is hereby given as a function of U, but a more complicated function than appears on the surface, since A depends upon B (equation 10) and B upon U (equation 11). Yet when we differentiate S_m with respect to U, and in so doing take account of these complications, it turns out that we might as well have been oblivious of them! for the result is the same as though A and B were constants:

$$dS_m/dU = kB \tag{23}$$

Now the temperature, which has so often slipped into this argument in ways more or less surreptitious, is about to make its formal and ceremonious entry into the statistical picture. We turn back to equation (17), and deduce:

$$dS/dU = 1/T \tag{24}$$

The derivative here standing on the left is the derivative of entropy with respect to energy under the condition of constant volume: a thermodynamicist would write it $(\partial S/\partial U)_v$. It is therefore properly to be identified with the derivative in (23), and we make the two identical by putting:

$$B = 1/kT \tag{25}$$

Now taking the entropy S to be the sum of S_c and S_m, we find:

$$S = S_c + S_m = -R \ln A + U/T + R \ln V - R \ln V_0 \tag{26}$$

and this is to be compared with (19), the thermodynamic expression for entropy, which I repeat to make the comparison easier:

$$S = \int (C_v/T) dT + R \ln V + \text{constant} \tag{27}$$

Comparing these, we see first of all that $R\,ln\,V$ appears in both, as was already stated. It also seems at first glance that $(-R\,ln\,A + U/T)$ is to be identified with the integral in (27), and that $-R\,ln\,V_0$ is to be identified with the constant in (27). This however is not necessarily the case, for $(-R\,ln\,A + U/T)$ may prove to include constant terms. Indeed they do; and we must proceed to evaluate both A and U in terms of T in order to round off the task.

I recall equation (10) and write it thus:

$$1/A = \Sigma \exp(-E_i/kT) \tag{28}$$

This is a summation, to which each cell contributes one term having the value of E appropriate to that cell—E_i for the ith cell. Of the volumes of these cells I have thus far said nothing, except that all are equal. I continue to say nothing further, but I give to their common volume the symbol H_0. Let us now form the integral:

$$\iiint \exp(-E/kT)\, dp_x dp_y dp_z, \qquad E = (1/2m)(p_x^2 + p_y^2 + p_z^2) \tag{29}$$

the range of integration extending over the whole of momentum-space. This integral may be described as follows. Let the momentum-space be divided into cells of unit volume. Each of these cells of unit volume makes a contribution

$$\exp(-\bar{E}/kT)$$

to the integral, $\bar{E}$ standing now for the average value of E in the cell in question. The integral is the sum of all of these contributions. Now let us inquire how much of a contribution is made by this same cell of unit volume to the summation (28). This second contribution is made up of $1/H_0$ terms, one for each of the cells of volume H_0 which occupy the cell of unit volume. The values E_i corresponding to these cells will not be exactly equal to the value $\bar{E}$ corresponding to the entire cell of unit volume; but to the degree of approximation which is now being used, the difference may be neglected. The summation (28) is then equal to $1/H_0$ times the integral (25). Now the value of the integral (29) is given in all tables of definite integrals, and in terms of our symbols it amounts to

$$(2\pi mkT)^{3/2}$$

so we come to the conclusion:

$$\begin{aligned} \ln A &= -\ln(2\pi mkT)^{3/2} + \ln H_0 \\ &= -\tfrac{3}{2}\ln T - \ln(2\pi mk)^{3/2} + \ln H_0 \end{aligned} \tag{30}$$

Now we have attended to every term in (26) except the term U/T. Nearly every reader will remember that the average kinetic energy of an atom of a monatomic gas at temperature T is $\frac{3}{2}kT$. I therefore leave out the derivation of this result, except for showing the student how to begin on it: the first step is to go back to equation (11) where an expression was given for $\bar{U}$, and in that expression to replace the summation $\Sigma\ E_i \exp(-BE_i)$ by $(1/H_0)$ times the integral $\iiint E \exp(-BE) dp_x\, dp_y\, dp_z$. It follows that U/T is $(3/2)\ Nk$, which for one gramme-molecule of gas is $(3/2)R$, which I write as $R \ln e^{3/2}$.

The picture of entropy for a monatomic gas limned by the Boltzmann statistics, is now completed. Entropy is the function which follows:

$$S = \frac{3}{2} R \ln T + R \ln V + R \ln \frac{(2\pi mke)^{3/2}}{V_0 H_0} \tag{31}$$

The dependence on volume is correct, *i.e.*, just the same as in the thermodynamic formula. The dependence on temperature is correct, for $(3/2)R$ is the value of the specific heat at constant volume per gramme-molecule of a gas, the quantity C_v of equation (18). The additive constant, as to the value of which thermodynamics says nothing, is fixed when the volumes V_0 and H_0 of the elementary cells in the ordinary space and the momentum-space are fixed.

Mixtures of Gases

Now we will go through the mental operation which is called "considering a mixture" of two different monatomic gases, N' atoms of the one and N'' atoms of the other, in the same box and (necessarily) in the same momentum-space. Let me denote by U' and U'', respectively, the energies of these two gases; and by $N_i' = N'w_i'$ and $N_i'' = Nw_i''$, respectively, the numbers of atoms of the two kinds in the ith cell of momentum-space.

If we seek the most probable distribution of the first gas in the momentum-space, making the stipulation that we will admit only such variations of the quantities w_i' as leave N' and U' unchanged—well, of course, we get the same result as before, the distribution (8), with N' in place of N and (let me say) A' in place of A and B' in place of B. A' will depend upon B' and B' will depend upon U'/N'. If we do the like with the second gas, we get anew to the distribution (8) with N'', A'' and B'' in place of N', A' and B'. A'' will not be the same as A' nor will B'' be the same as B', unless it happens that U''/N'' is equal to U'/N'. There is no cause for surprise in this. In acting this way we are only treating each gas by itself, and have as yet done nothing which can be regarded as "considering a mixture".

Let us however seek the most probable distribution of the two gases, mak-

ing the stipulation that we will admit only such variations of the quantities w_i' and w_i'' as leave N' and N'' and *the sum of the energies U' and U''*—not however the individual energies U' and U''—unchanged. In acting this way we *are* doing something which may be regarded as "considering a mixture", since we are allowing for the possibility that energy may pass from the one gas to the other and the other to the one. Equally well are we considering the case of two gases separated by a partition through which energy may pass, but not the atoms. Since in such a case we really ought to take into account the atoms and the energy of the partition also, we must appease the critics by providing that the partition shall be very thin.

Choose any set of values of the quantities N_i', which is to say, any particular distribution of the first gas; and choose any set of values of the quantities N_i'', which is to say, any particular distribution of the second gas. Go back to equation (1) and put primes on all the symbols N, N_1, N_2, $\cdots$ on the right-hand side of that equation. The resulting expression gives the total number of inventories or complexions of the first gas. Take off the primes and affix double primes to each of these symbols. The resulting expression gives the total number of inventories or complexions of the second gas. Every complexion of either may coexist with any complexion of the other. Therefore the total number of complexions of the pair of gases is the product of the two expressions. It is this product which is W for the pair of gases, be they mixed or side-by-side.

With use of the Stirling approximation, the logarithm of W for the pair is the sum of two such expressions as we have seen in (5):

$$\ln W = -N'\Sigma w_i' \ln w_i' - N''\Sigma w_i'' \ln w_i'' \tag{32}$$

and its variation is:

$$\delta \ln W = -N'\Sigma(1 + \ln w_i)\delta w_i - N''\Sigma(1 + \ln w'')\delta w_i'' \tag{33}$$

Let us now give a trial to the tentative distribution,

$$w_i' = A' \exp(-B'E_i), \qquad w'' = A'' \exp(-B''E_i) \tag{34}$$

On substituting this into (33) we find that if B' is unequal to B'', the distribution has a stationary value of W with respect only to such variations as leave the energies of the two gases separately unchanged—the result which we had before. If however B' and B'' are the same, then W is stationary with respect to variations which leave the sum of the energies unchanged, either being allowed to gain or lose so long as the other loses or gains by an equal amount. Since each B is controlled by the corresponding U/N, the distribution (33) has a stationary value of W for variations of the type in question if and only if the average energy of the atoms of each gas is the same. Since each B controls the corresponding A, this condition of

equal average energy makes the distributions of the two gases just the same.

We have already seen that kB is the reciprocal of the temperature: for it is the reciprocal of $(\partial U/\partial S)_v$ in our statistical picture, and the definition of absolute temperature T is precisely that T is this derivative. The statement to which we have come is, that *the most probable state of the mixture is the one in which T is the same for both components*. It is often expressed in this way: classical statistics shows that for two (or more) gases in equilibrium with each other, the temperature must be the same. It is indeed a fact of experience, and a most important one, that when two systems (be they gases or be they not) are in thermal equilibrium, their temperatures *are* the same. This has not hitherto been mentioned, and yet we seem to have derived it. Quite a rabbit for the magician of the classical statistics to have pulled out of the hat!

However, skeptical people who see a rabbit pulled out of a hat are inclined suspect that either the rabbit was in the hat beforehand, or else there is no rabbit. Let us inquire into the contents of the hat and see whether we can find the rabbit there.

The first (and the last) question to be asked is: what *is* the difference between "different" kinds of gas in the statistical picture?

To the physicist or the chemist, different kinds of gas will be (for example) mercury and helium. These differ in their spectra, boiling-points, chemical properties, and quantities of other features. None of these features however appears in the theory, and therefore none of them can contribute to the result. The atoms also differ in mass, and for a moment this seems to be a difference of which the statistical picture takes account, since the letter m appears in some of our equations. However, it appears only in the ultimate equations, those such as (29) in which the distribution-in-momentum is expressed. It does not appear in the original form of the Maxwell-Boltzmann distribution-in-energy, the form shown in equation (8). It appears in particular in the last term of equation (31), but not elsewhere. Apart from this it may be said that *in the classical statistics, all gases are the same gas.*

This is a paradox, but only one of two. The other paradox is, that *in the classical statistics two parts of the same gas are different gases.* This second paradox arises from the numbering of the molecules, which is an essential feature of the classical statistics.

Therefore in the statistical picture a mixture of N' atoms of mercury and N'' atoms of helium is distinguished by the fact that the mercury atoms bear one set of integer numbers (say those from 1 to N') and the helium atoms another set (say those from $N' + 1$ to $N' + N''$). But if the atoms were all helium atoms or all mercury atoms, they would also be divisible in many

different ways into a set of N' atoms bearing one set of numbers and a set of N'' atoms bearing another set of numbers. Each set would obviously have to have the same distribution, with the same A and the same B, as any other set or as the totality of all the atoms. This conclusion, which is self-evident in the case in which all the atoms are called "mercury", remains true when some of the atoms are called "mercury" and others are called "helium". We have done nothing but change the names of some of the atoms; we have not imported into our theory anything which differentiates one kind of atom from another kind. No wonder we have arrived at the conclusion that all kinds have the same distribution-in-energy, the same A, the same B and the same temperature! The rabbit was indeed in the hat, but it does not look like so much of a rabbit.

The classical statistics therefore doesn't recognize any of the real differences between atoms of different kinds, except for alterations in the last term of (31); but it does make an artificial difference which creates the astonishing result, that any two samples of the same gas are different gases! At this point we may begin to wonder whether this peculiarity, which has led to so apparently brilliant a result in respect of the equality of temperatures in thermal equilibrium, might elsewhere lead us astray. It does; and here appears the rift in the lute of classical statistics.

The Rift in the Lute

Let us imagine two boxes of equal size separated by a common partition, each containing a gas consisting of N atoms, both gases at the same temperature. We will baptize one gas "mercury" and the other gas "helium". Let an opening be made through the partition. It is known that in such a situation in Nature, the two gases diffuse into one another, the final and permanent condition being that in which the mercury and the helium are equally distributed between the two boxes. The process of diffusion is an example of what in thermodynamics is called an "irreversible" process. The state of uniform mixing ought to correspond to the most probable state in the statistical picture. But what does the statistical theory say?

The statistical theory says nothing about diffusion and nothing about mixing. The statistical theory takes account of nothing but the facts that the mercury had at its disposal the volume V before and the volume $2V$ after the breaking of the partition, and ditto for the helium. The value V contains M cells ($M = V/V_0$) and the volume $2V$ contains $2M$ cells. The (approximate) probabilities of the uniform distribution are M^N before and $(2M)^N$ after. The latter is greater than the former; the entropy goes up by $Nk \ln 2$ for each gas, by $2\ Nk \ln 2$ for the two of them, when the private preserve of each is thrown open to the other. This gain is what is called the

"entropy of mixing" though as we have seen it is really the "entropy of expansion". It is the alteration in the second term of the righthand member of (31).

But now suppose both of the boxes hold helium. One may indeed continue to suppose that when the partition is opened each one of the two samples of helium undergoes an expansion, doubling its volume. The entropy would then go up by $2\,Nk \ln 2$. However this looks so silly a thing to say that no one, I feel almost secure in affirming, has ever said it. The natural thing to say is, that the $2N$ atoms of helium distributed through the two boxes at uniform temperature and uniform pressure have just the same entropy-value whether or not the partition is broken.

What does the classical statistics say about this situation? Its answer can be foretold. Since the two samples of helium are different by virtue of the different numberings of the two sets of atoms, the classical statistics insists that the entropy increase by $2Nk\,ln\,2$ when the partition is broken, even though the gases are the same. This is indeed, if I may pervert the poem, "the little rift within the lute, which makes the classical statistics mute." The achievement of predicting the uniform distribution in ordinary space, the achievement of predicting the Maxwell-Boltzmann distribution-law in momentum-space, the achievement of providing the proper relation between temperature and mean kinetic energy—all of these are unsettled by this calamity.

Were I writing a strictly logical article I should quit at this point. Nothing further can apparently be done, except to tamper with the classical statistics in an effort to remove the unwanted result which has sprung forth to plague us. To violate the logic of the classical statistics in order to banish the undesired while keeping the desired results is a very questionable act. In theoretical physics, it is not admissible that the end justifies any and all means. Nevertheless so successful a feat of tampering has been done, that I cannot refrain from mentioning it as I close.

Let me first express in a slightly different way the nature of the "rift". Compare two samples of the same gas at the same temperature, one consisting of N atoms in a volume V, the other consisting of xN atoms in a volume xV. That which is called entropy in thermodynamics—and therefore that which *is* entropy, since it is the privilege of thermodynamics to give the definition of entropy—is x times as great for the latter as for the former. But that which the classical statistics calls entropy—or, as we must admit, miscalls entropy—is not x times as great for the latter as for the former. It would be, if there were x times as many atoms but just the same number of cells. However, there are x times as many atoms but also x times as many cells into which to put them. The number of complexions is approximately M^N in the former case and $(xM)^{xN}$ in the latter, M standing

for the number of cells in the former box. The thing miscalled entropy is $kN\ln M$ in the first case and $(kxN\ln M + kxN\ln x)$ in the second case. It is the term $kxN\ln x$ which is the rift.

Clearly we could abolish this term by allowing the volume of the cells to swell in the ratio x:1 when going from the former case to the latter. This is the same as making H_0 proportional to the number of atoms in the sample of gas which happens to be under study. Since in equation (31) the volumes V_0 and H_0 (of the elementary cells in ordinary space and in momentum-space) are indissolubly bound together in the product V_0H_0, this is the same as making V_0H_0 equal to some constant multiplied by the number of atoms under study.

Such, if I interpret correctly, was the idea proposed by Sackur in 1912. While it does the task required, it is an "*ad hoc*" assumption of the most barefaced character. If the gas under study is at first divided into two parts by a partition and the partition is then abolished, the cells must be supposed to swell up at the moment when the partition vanishes.

We can also abolish the fatal term by going back to equation (1) for the number of complexions, and removing the factor $N!$ in the numerator and replacing it by unity. We then have unity divided by the original denominator, which in the (most probable) case of the uniform distribution is $(N/M)!$ raised to the power M, as I remarked on page 121. Using the super-Stirling approximation, we find that the logarithm of one fraction is $(N\ln M - N\ln N)$. The factor $N!$ which we formerly had in the numerator killed off the term $(-N\ln N)$, but now that we have taken it out, this term survives. If now we say that k times the logarithm of $W/N!$ shall be the picture of entropy in the classical statistics, then the term $(-kN\ln N)$ comes over into the right-hand member of (31). It may be amalgamated with the last term already standing there; and when this is done, we find V_0H_0 multiplied by N exactly as Sackur put it there, and with the same wished-for result.

This, if I interpret correctly, is the idea proposed in 1913 by Tetrode. It does the task required of it, but its drawback is that the removal of the factor $N!$ from the right-hand member of (1), a drastic piece of surgery as it were, violates the system of the classical statistics.[3]

I was not, however, thinking merely of this achievement when on Page 132 I spoke of "a remarkably successful feat of tampering." To show the

[3] This may seem too strong a statement. We are, after all, only asked to accept k ln $(W/N!)$ as our picture of entropy, instead of $k\ln W$; why be reluctant? But in effect, as I see it, we are asked first to accept $k\ln Wf$ as our picture of entropy, f being an arbitrary function of N; and then we are asked so to choose f, that the dependence of k ln Wf on N shall conform to the actual behavior of entropy. This is different from and much less impressive than our original procedure, which consisted in first realizing that W is the number of complexions, and then discovering that k ln W depends on volume and on temperature in just the right ways for entropy.

magnitude of the achievement, I will rewrite equation (31) with two alterations. The first consists in replacing R with Nk, so that the expression shall refer not to a gramme-molecule of gas but to any number N of atoms. The second consists in following Tetrode by affirming that the entropy is not $k \ln W$, but k times the logarithm of $W/N!$ I follow him still further by using, not the super-Stirling approximation in which $N \ln N$ is written for $\ln N!$, but the better approximation in which $(N \ln N - N)$ or $(N \ln N - N \ln e)$ is written for $\ln N!$ The result is:

$$S = (3/2)Nk \ln T + Nk \ln V - Nk \ln N + Nk \ln [(2\pi m k^{3/2} e^{5/2}/V_0 H_0] \quad (35)$$

This quantity newly chosen as the picture of "entropy" depends on volume and on temperature in the right way, as did the other. The dependence on N the number of atoms is now correct, and no wonder, for the new quantity was chosen with that purpose. There is a fourth term in the right-hand member which is proportional to N, and its value is completely determined if the value of $V_0 H_0$ is fixed. The value which it takes when N is made equal to N_0 may be called "the chemical constant"; but this name has been spoiled through being used with several different meanings, and should probably be abandoned.

When to $V_0 H_0$, the volume of the elementary cell in six-dimensional space, there is given the value h^3—the cube of Planck's constant—the resulting value of the fourth term is excellently confirmed by experiments on all of the noble gases, and (with less precision) by experiments on many of the monatomic vapors of metallic elements. This is the achievement known as "the verification of the Sackur-Tetrode formula" and it is indeed a grand one.

Anyone versed in thermodynamics will probably regard this not as a grand result, but as an incomprehensible one! Are we not taught in thermodynamics that nothing is ever measured about entropy except the *differences* between its values under different conditions, so an additive constant like the one in question must drop out of every verifiable equation, and its value can never be found? How then can it make sense to speak of confirming the value of the fourth term on the right-hand side of (35)?

Well, actually it *is* a difference which is measured: the difference between the entropy of the gas at any convenient temperature and volume and the entropy of its solidified crystalline form at the absolute zero. This difference is found to be such, that if for the entropy of the gas one puts the value (35) with h^3 substituted for $V_0 H_0$, then for the entropy of the crystalline solid at the absolute zero one finds the value: *zero*. This result—this conclusion that the entropy of a crystal is zero at the absolute zero—is in itself so desirable and welcome that it is taken as the confirmation of the

Sackur-Tetrode formula. By "desirable and welcome" I mean that it is harmonious with the idea that *entropy is a measure of disorder*, an idea plausible in itself and fruitful in its applications. A chemical element perfectly crystallized at the absolute zero is supposed to be the exemplar of supreme order, and therefore its entropy ought to be nil. But this is an enormous subject requiring at least one other article, and I am glad that my attempt at writing such an article stands already in print in the June(1942) issue of this Journal.

Here then is the astonishing history of the Classical Statistics. By a strangely artificial device, the numbering of atoms deemed identical, it arrived at the proper distributions—that is, the distributions ratified by experiment—in ordinary space and in momentum-space. It then proposed a picture of entropy partially right, yet wrong in its dependence on the number of atoms, and therefore fatally wrong. With another artificial and dubious device, it corrected itself by adopting a new picture of entropy, this time depending in the right way upon the number of atoms. With a third artificial device (the introduction of Planck's constant in a peculiar way) it completed the formula for entropy in a manner leading to the consequence that the entropies of solidified crystallized elements are zero at absolute zero. All of these feats and more were subsequently achieved by the New Statistics, in a manner which I hope to explore on a later occasion.

Abstracts of Technical Articles by Bell System Authors

Poles and Pole Treatment.[1] REGINALD H. COLLEY. Studies made of pole use and drainage on the southern pine forest have brought out that if the demand for poles 35-feet and longer were to continue at the present rate, a situation would soon develop in which these poles would be at a premium. It would seem wise to use as many circumference classes as possible and to broaden the use of poles of other species whenever it is practicable to do so. Successful full length treatment of red pine, lodgepole pine, western cedar and Douglas fir poles will help to broaden this use.

There has been a definite trend toward greater mechanization in pole production. Machine shaving smooths the pole surface and accelerates drying. Poles with square cut roofs and slab gains are all-purpose poles, one design taking the place of four. New types of preservative treatment—greensalt, creosote-petroleum-pentachlorphenol and salt-creosote—are of promise and must be considered, when current restrictions are removed, in those cases where clean poles are mandatory.

Interesting breaking test data are reported which show that the modulus of rupture of pole top sections average 90 per cent of the modulus of the poles as a whole, and that pole tops are sufficiently strong to meet their specified class breaking loads. This is of considerable importance where poles are guyed.

Current groundline treatment methods, it is pointed out, should be applied to untreated poles in line that are worth saving at the time of inspection, as a part of the regular inspection procedure.

Hearing, the Determining Factor for High-Fidelity Transmission.[2] HARVEY FLETCHER. This paper gives the requirements for ideal systems for the transmission of speech and music. These requirements are based on: 1. Measurements of the threshold and frequency limits of the hearing of more than 500,000 people at the New York and San Francisco World's Fairs; 2. measurements of the discomfort level of sound; 3. measurements of room noise in a wide variety of locations; and 4. measurements of the frequency limits and the maximum and minimum levels of speech, orchestral music, and various instruments of the orchestra.

From this information and from judgment tests it is concluded that substantially complete fidelity in the transmission of orchestral music is

[1] *Elec'l. Engg.*, Transactions Section, September 1942.
[2] *Proc. I. R. E.*, June 1942.

obtained by use of a system having a volume range of 65 decibels and a frequency range from 60 to 8000 cycles per second. Substantially complete fidelity for the transmission of speech is obtained by a system having a frequency range from 100 to 7000 cycles per second and a volume range of 40 decibels.

Preliminary experiments comparing a single-channel system and a two-channel stereophonic (auditory perspective) system showed that stereophonic transmission with an upper frequency limit of 5000 cycles per second was preferred to single-channel transmission with an upper limit of 15,000 cycles per second. A definite improvement was obtained in the stereophonic system by using three channels instead of two.

A New Direct Crystal-Controlled Oscillator for Ultra-Short-Wave Frequencies.[3] W. P. MASON and I. E. FAIR. An ultra-high-frequency crystal oscillator is described which utilizes a mechanical harmonic of an AT or BT crystal. With the oscillator frequencies as high as 197 megacycles, hormonics as high as the 23rd have been excited. Taking the second electrical harmonic of the oscillator, frequencies as high as 300 megacycles, or 1 meter have been obtained. Since a mechanical harmonic is used, the crystal can be of a practical size to handle and adjust. The harmonic vibration of the AT and BT crystals have as low a temperature coefficient as the fundamental mode, and temperature coefficients of less than two parts per million per degree centigrade are easily obtained. Stability curves for this type of oscillator are shown and the results indicate that at 120 megacycles stabilities in the same order of magnitude as for ordinary crystal oscillators can be obtained. Without temperature or voltage control it appears likely that the frequency should remain constant to $\pm$ 0.0025 per cent.

Some measurements have been made of the properties of harmonic crystals at high frequencies. It was found that the Q of a crystal is independent of the frequency but in general increases with harmonic order. The ratio of capacitances r of a crystal increases as the square of the harmonic order. It is shown that in order to obtain a positive reactance in the crystal $Q > 2r$. This relation will only be satisfied for harmonics of AT crystals less than the 7th. As a result oscillator circuits such as the Pierce circuit cannot be used to drive crystals at high harmonic frequencies. A discussion of oscillator circuits is given and it is shown that a capacitance-bridge oscillator circuit with the crystal in one arm is the best type to use for high-frequency harmonic crystals.

War Activities of the Bell Telephone System.[4] KEITH S. MCHUGH. The scope of the Bell System's nation-wide service is, even in peace, difficult to

[3] *Proc. I. R. E.*, October 1942.
[4] *Bell Telephone Magazine*, November 1942.

visualize in its entirety. In war, when practically every phase of the national effort to overthrow the Axis aggressors depends in some part on swift communication, both the extent and the importance of the System's contributions to the winning of the conflict are beyond summarizing. In the past two years, numerous articles in the *Bell Telephone Magazine* (listed at the end of this article), and in the employee publications of the Associated Companies, have described many aspects of the System's cooperation with the armed forces, with industry, and with the civilian population. Now, nearly a year after Pearl Harbor, it seems appropriate to review both the System's preparations for the national emergency and the steps which it has taken since war became no longer a threat but a fact. To the extent that it is possible in limited space, this article rounds out the previous fragmentary parts of the whole picture.

The Number of Two-Terminal Series-Parallel Networks.[5] JOHN RIORDAN and C. E. SHANNON. This paper is concerned with the number of ways n abstract (electrical) elements may be connected in series-parallel arrangements and in particular with the way the number behaves for n large. After a proof of a generating identity for the numbers given without proof by P. A. MacMahon in 1892, the paper gives recurrences and schemes of computation by means of which MacMahon's table for the numbers is extended from $n = 10$ to $n = 30$. The behaviour for n large is shown to be of the form

$$A\,\lambda^n\,n^{-3/2}$$

with A a fixed constant and λ a real number between $2 + \sqrt{2} = 3.414$ and 4 and closer to the former than the latter; indeed an approximating function for which λ is about 3.56 agrees with the numbers within 3% over the range 7 to 20. These results are used to show that almost all switching functions of n variables require at least

$$(1 - \epsilon)\,\frac{2^n}{\log_2 n} \qquad \epsilon > 0$$

switching elements (make or break contacts) in series-parallel realization.

The Electrical Oscillations of a Perfectly Conducting Prolate Spheroid.[6] ROBERT M. RYDER. The forced oscillations of a perfectly-conducting prolate spheroid of eccentricity nearly unity are shown to be decomposable into "harmonics" corresponding to different modes of vibration, each harmonic being quantitatively connected with a certain portion of the impressed electric field which drives the antenna. The harmonics contribute additively to the current and field of the spheroid; each offers a characteristic imped-

[5] *Jour. Mathematics and Physics*, August 1942.
[6] *Jour. Applied Physics*, May 1942.

ance to the driving field, and the properties of the antenna are a composite depending upon the proportions of the various harmonics present. The behavior of the harmonics with frequency is discussed qualitatively; analytical expressions obtained are useful chiefly at the resonant frequencies of the antenna, where the most important harmonic becomes sinusoidal in character.

On Radiation from Antennas.[7] S. A. SCHELKUNOFF and C. B. FELDMAN. This paper presents some theoretical remarks and experimental data relating to applications of the transmission-line theory to antennas. It is emphasized that the voltage, the current, and the charge are affected by radiation in different ways, a fact which should be considered in any adaptation of line equations to antennas.

It is shown experimentally and theoretically that in an antenna of length equal to an integral number of half wave-lengths, which is energized at a current antinode, the effect of radiation on the current and the charge (but not on the voltage) can roughly be represented by adding to the resistance of the wires another fairly simple term.

The Use of Secondary Electron Emission to Obtain Trigger or Relay Action.[8] A. M. SKELLETT. The use of secondary electrons to obtain trigger action similar to that of a thyratron is described. An experimental tube and the necessary circuits by which this action is achieved are discussed. This combination gives the features of a triode with a relay or on and off feature, resulting in an amplifier, oscillator, modulator, or other vacuum tube device which may be turned on or off abruptly at high or low frequencies. In addition, it can be used to replace thyratrons in many of their circuits where very low impedance is not necessary and is capable of much greater speeds of operation in such applications.

A New Frequency-Modulation Broadcasting Transmitter.[9] A. A. SKENE and N. C. OLMSTEAD. A new frequency-modulation transmitter is described which uses a novel amplifier circuit permitting an unusually simple mechanical design and an economical vacuum-tube complement.

The choice and design of circuit components, governed by both mechanical and electrical considerations, are discussed in detail.

A Secondary Frequency Standard Using Regenerative Frequency-Dividing Circuits.[10] F. R. STANSEL. A secondary frequency standard is described

[7] *Proc. I. R. E.*, November 1942.
[8] *Jour. Applied Physics*, August 1942.
[9] *Proc. I. R. E.*, July 1942.
[10] *Proc. I. R. E.*, April 1942.

in which standard frequencies are derived from a 5-megacycle oscillator by a series of frequency dividers. The advantage of obtaining standard frequencies by frequency division rather than by frequency multiplication is pointed out and the characteristics of the regenerative frequency dividers used are discussed.

Some Mechanical Aspects of Telephone Apparatus.[11] J. D. TEBO and H. G. MEHLHOUSE. *Part I.* It is seldom realized that the vastness of the Bell System requires such an enormous amount of equipment and wires for handling the 100,000,000 calls per day made by the people in the United States. A total of 44,000 kinds of apparatus involving 170,000 different parts are required. The crossbar switch, described in this article, is the specific telephone switching mechanism used in the latest dial telephone system.

Of particular interest in the crossbar switch is the design of contact springs. These springs are essentially thin, metal beams of a rectangular cross section but of varying cross sectional area along their length. They are clamped at one end and are subject to bending as compound cantilevers. Since the clamping is necessarily between relatively soft insulating materials, the determination of the effective length of the spring in determining its true deflection curve required the use of unique methods. Likewise, since the springs are punched out of sheet stock at an angle to the grain direction of the material, the modulus of elasticity does not remain the same for equal cross sections of the same material. Consideration of these points was necessary in determining the strength of the magnets for operating the contact springs, as well as to insure that the stresses introduced in the springs would not be excessive.

To study the motions of the various parts of the switch, both high speed motion pictures and the "rapid record" oscillograph were used. The oscillograph was provided with means for obtaining "shadowgrams" of the actual movement of parts simultaneously with the changes in the electrical characteristics of the magnets and contacts. The use of both the camera and oscillograph provided valuable data for making improvements, both in design and operating characteristics.

Part II. The manufacture of crossbar apparatus is accomplished on a product basis; that is, the entire range of manufacturing operations is segregated into one division, and practically all operations from raw material to the completed product are performed in this division. More than 150 kinds of parts totaling an annual demand of 200,000,000 individual pieces are required to produce the crossbar switches.

To produce these parts requires a number of special machines, tools, and

[11] *Mech. Engg.*, May 1942 and June 1942.

operations. Of particular interest are the use of special progressive punch and die tools, coil winding, contact welding and conveyorized assembly and adjustment.

The use of heavy presses with large progressive punches and dies was necessitated by the degree of accuracy required for this grade of equipment. For example, the vertical unit base, weighing only $\frac{3}{4}$ lb., is produced by a 75,000 lb. press, using a 3,500 lb. tool. Again, since 40,000,000 contact springs are required annually, each of which must be attached to insulators, automatic presses, conveyor belts and handling devices are required. Welding two contacts on each spring is accomplished by special welding presses using rolls of contact metal tape, each contact being cut off just prior to the welding process. A quality of less than one defective contact out of 20,000 is maintained.

The coils are wound in special machines in "sticks" of 5 to 7 coils with .0007 inch thick cellulose acetate between each layer. A wire of a gage size halfway between # 37 and # 38 B & S is used on most of the coils because of critical capability conditions—the annual amount of wire of this size being 125,000 lbs.

Assembly and adjustment of the switches proceeds on conveyor belts from one end of a large room to the other—the procedure being set up in such a way as to create a continuous flow of completed parts for wiring into the large frames ready for installation in telephone exchanges.

Regulated Rectifiers in Telephone Offices.[12] D. E. Trucksess. For many years rectifiers of the garage type were used in converting alternating current to direct current for charging batteries used for communication purposes. These batteries furnish power for relay operation, for talking, and filament and plate supplies for repeaters. The rectifiers were of the manual-control type where the operator selected the charging current by means of tap switches or rheostats.

With the development of the thyratron type of tube, a rectifying means was made available in which the grid of the rectifier tube could be used to control its own output current by an electronic circuit. Rectifier circuits were designed to maintain a constant output voltage. If a regulated rectifier is connected to a battery and the constant rectifier voltage is 2.15 volts per cell, the load current will automatically come from the rectifier and not from the battery. Also the battery will draw from the rectifier sufficient additional current to maintain its charge. If the circuit voltage is held within limits of less than plus or minus one per cent, the maintenance of the battery is reduced and its life is extended.

The thyratron tube differs from the vacuum tube in that the grid does

[12] *Elec'l. Engg.*, Transactions Section, August 1942.

not have a continuous control of the plate current. When a positive potential is applied to the plate, current does not flow until the magnitude of the negative grid voltage is reduced to the critical value, at which time the plate current flows, and the magnitude of the plate current depends upon the voltages and impedances in the circuit. The grid has no further control, and plate current flows until it is stopped by reducing the plate voltage to zero.

Thyratron tubes use various gases and mixtures of gases. The earliest type used mercury vapor, but this type of tube is quite sensitive to temperature changes. The grid characteristics are shifted materially by changes in the room temperature in which it is operated, and in low temperatures it is almost a vacuum tube. Thyratron tubes using argon gas are not affected by temperature changes, but high-pressure argon tubes have a low inverse voltage which limits their application to low-voltage rectifiers. Tubes using low-pressure argon have a higher inverse voltage, but are accompanied by a high arc drop which makes their efficiency low. A mixture of mercury vapor and argon has been found which provides the temperature-stable grid characteristic of the argon tube and the low arc drop of the mercury-vapor tube. This type of tube has been very successful with certain regulating circuits, particularly at voltages less than 60 volts.

Five kinds of regulating circuits are used in telephone offices to hold the output voltage of rectifiers constant. The selection of the circuit to be used depends upon the magnitude of the current, d-c voltage, and type of rectifying means to be used. Two forms of regulating circuits using thyratron tubes and one using two-element high-pressure tubes were developed. A fourth circuit using all vacuum tubes was adapted for telephone use. The fifth kind uses a negative resistance.

In this paper a table shows the voltage and current output, type of control of the rectifiers, and the rectifying means that have found widespread use in the Bell System.

The regulated rectifier finds its applications in telephone offices where constant voltage, independent of load and a-c line-voltage variations, is required to supply filament grid bias and plate voltage to telephone repeaters. Certain measuring circuits require a regulated rectifier to supply a stabilized voltage. Regulated rectifiers also find applications where constant voltage is of secondary importance but an automatic power plant is desired for maintaining storage batteries in a fully charged condition to be ready to supply the power for telephone offices if the a-c power fails. A further compensation of regulating the voltage is the increase in life obtained from storage batteries if they are not continually being charged and discharged but are fully floated.

Contributors to this Issue

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For the purposes of record and assistance to librarians, and for the information of subscribers, it is to be noted that there was no April 1943 issue of the Bell System Technical Journal.

It is also to be noted that there was only one issue of Volume 21 of the Bell System Technical Journal. It was the issue of June 1942.